THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The two hundred twenty-first regular meeting of the Society was held at Columbia University on Saturday, February 25, extending through the usual morning and afternoon sessions. The attendance included the following seventy-five members:

Alexander, Beal, Birkhoff, Borden, Bowden, E. W. Brown, T. H. Brown, E. T. Browne, B. H. Camp, G. A. Campbell, Cole, Coolidge, Cowley, Cronin, Dantzig, Douglas, Eisenhart, Fine, Fischer, Fite, Philip Franklin, Fry, Gafafer, Gilman, W. C. Graustein, Gronwall, Grove, Haskins, Hausle, Hebbert, Robert Henderson, Hitchcock, Huntington, Dunham Jackson, Joffe, Kellogg, Kline, Lamson, Leavens, J. L. Love, McDonnell, McMackin, MacDuffee, J. B. Maclean, MacNeish, Mathews, H. H. Mitchell, Moir, Frank Morley, Mullins, Northcott, Oglesby, Osgood, Pfeiffer, Post, Press, Reddick, Reid, R. G. D. Richardson, Ritt, Ruger, Safford, Seely, P. F. Smith, Sosnow, Strong, Stuerm, Tanoue, J. S. Thompson, Wedderburn, Weisner, M. E. Wells, H. S. White, R. G. Wood, J. W. Young.

Thirty-five persons were elected to membership by the Council at this meeting as follows:

Dr. Anne Dale Biddle Andrews, University of California;
Professor Lloyd Collier Bagby, Ottawa University;
Professor Apolinario Baltazar, University of the Philippines;
Dr. John Douglas Barter, University of California;
Mr. John Biggerstaff, Reed College;
Professor Augustus Bogard, College of St. Teresa;
Mr. Garland Baird Briggs, University of Virginia;
Professor Frank E. Burcham, Central College;
Professor William Russell Burwell, University of Tennessee;
Mr. José Caminero, Havana;
Mr. Wendell Morris Coates, University of Michigan;
Professor Harold Athelstane Fales, Columbia University;
Professor Maurice Fréchet, University of Strasbourg;
Mr. Carl Arshag Garabedian, Harvard University;
Mrs. Eleanor Growe, University of California;
Professor Napoleon Bonaparte Heller, Temple University;
Professor Frank Kelso Hyatt, Pennsylvania Military College;
Professor Jesse Bythwood Jackson, University of South Carolina;
Professor Miles Abdel Keasey, Drexel Institute;
Professor Joseph Patrick Kelly, Boston College;
Mr. Arthur Louis Klein, California Institute of Technology;
Dr. Sophia Hazel Levy, University of California;

Mr. David A. Lunden, Cooper Union;
Professor Daniel Joseph Lynch, Boston College;
Professor James Thomas McCormick, Boston College;
Professor Patrick Joseph McHugh, Boston College;
Professor Pius Stephen Pretz, St. Benedict's College;
Miss Constance Rummons, University of Nebraska;
Professor Paul Gerhard Schmidt, St. Olaf College;
Professor William Henry Schuerman, Vanderbilt University;
Dr. Charles Donald Shane, University of California;
Professor George Waddel Snedecor, Iowa State College;
Miss Elizabeth Webb Wilson, Washington, D. C.;
Mr. Bing Chin Wong, University of California;
Professor Frank George Wren, Tufts College.

The Secretary also announced the election to membership, on January 16 (by special provision of the Council, under the suspension of By-Law I), of the following twenty-seven persons:

Mr. Rodney Whittemore Babcock, University of Wisconsin;
Mr. Raymond Walter Barnard, University of Michigan;
Professor Lawrence Henry Bowen, Furman University;
Mr. Martin Allen Brumbaugh, University of Pennsylvania;
Miss Elizabeth Carlson, University of Minnesota;
Professor Henry Clyde Carver, University of Michigan;
Miss Margaret Francis Chapman, Milwaukee State Normal School;
Professor Earl Church, Pennsylvania Military College;
Professor Charles Neblett Dickinson, Hollins College;
President Frances Augustine Driscoll, Villanova College;
Dean Robert Ryland Fleet, William Jewell College;
Mr. William Rogers Herod, General Electric Company, Schenectady;
Mr. Hemphill Hosford, Southern Methodist University;
Professor George Harold Jamison, Missouri State Teachers College, Kirksville;
Mr. Bradford Fisher Kimball, Harvard University;
Professor Paul Victor Levain, Holy Cross College;
Mr. James Lee Love, Gastonia Cotton Manufacturing Company;
Professor Ross W. Marriott, Swarthmore College;
Professor Ernest Alanson Pattengill, Iowa State College;
Professor William Richard Ransom, Tufts College;
Miss Margaret A. Shelley, College of St. Catherine;
Professor Hugo Ferdinand Sloctemyer, St. Louis University;
Mr. Sinclair Smith, California Institute of Technology;
Professor Arthur Dodd Snyder, Union College;
Professor Frederick Joseph Taylor, College of St. Thomas;
Mr. Wilmer Nichols Thompson, Drury College;
Professor Dudley Weldon Woodard, Howard University.

At the meeting of the Council, a letter was read from Professor E. V. Huntington reporting that an anonymous donor

had offered to give the sum of $4,000 to print an additional volume of the TRANSACTIONS in 1922; the Council accepted the offer, with thanks to the donor and the intermediary. The Society also adopted a resolution of thanks for this very generous gift when it was announced by the Secretary at the morning session.

The Council appointed a committee consisting of Professors P. F. Smith (chairman), Eisenhart, and Hedrick, with power to arrange for the printing of the Society's publications; this committee was in particular authorized to proceed with the immediate publication of the additional volume of the TRANSACTIONS.

Professor C. N. Haskins was elected to succeed Professor L. E. Dickson as representative of the Society on the National Research Council for a period of three years beginning July 1, 1922.

On recommendation of the Council, the Society voted to extend the suspension of By-Law I until the end of April, and also to adopt the following substitute for Section 3 of By-Law II:

"Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with the principle that the life membership be regarded as the present worth of a life annuity due of a yearly payment equal to the net value of the annual dues, the annuity to be based on McClintock's 4 per cent *Tables for Annuitants (Male Lives)*."

Professor H. B. Fine presided at the morning session, relieved in the afternoon by Professors C. N. Haskins and H. S. White. The afternoon session was especially marked by a paper read, at the request of the programme committee, by Professor J. L. Coolidge, on *The basis of mathematical probability*. A number of members of the Actuarial Society were present, by invitation, to hear this paper.

Titles and abstracts of the papers read at this meeting follow below. Mr. Langer's paper was read by Professor Birkhoff, and Professor Schwatt's two papers by Professor

H. H. Mitchell. The papers of Professor Kellogg, Dr. Walsh, and Professors Emch, Morenus, and Hollcroft were read by title.

1. Professors G. D. Birkhoff and O. D. Kellogg: *Invariant points in function space.*

In this paper, the authors announce some of their results on the problem communicated to the Society on Dec. 30, 1920 (this BULLETIN, vol. 27 (1921), p. 307). The existence is established of invariant points in a region of n-space which is convex toward an interior point, under a continuous, one-valued transformation which carries points of the region into points of the region; also of the inverses of points on the hypersphere in n-space (n odd) with respect to a parametric transformation containing the identity. A number of analogous theorems are thus inferred for function space, first by a method of interpolation, and second, by a transition through a Hilbert space. Some theorems on linear transformations are derived directly. Applications are given to existence theorems for differential equations and integral equations, non-linear and linear. Incidentally, a simple proof is given of the important theorem that the general algebraic m-spread in n-space is non-singular, of which the authors have not found a demonstration in the literature.

2. Professor O. D. Kellogg: *A property of certain functions whose Sturmian developments do not terminate.*

In this note, the author reports the generalization to developments in terms of solutions of differential equations of the Sturm type of a result previously announced for Fourier series (see this BULLETIN, vol. 18 (1911), p. 234). A theorem on the rate of growth of the maximum of the absolute value of the nth derivative of an analytic function, on a circle interior to a region of analyticity, as a function of n, is also given.

3. Professor G. D. Birkhoff and Mr. R. E. Langer: *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order.*

This paper develops the theory of a system of n ordinary linear differential equations of the first order containing a parameter and subject to certain boundary conditions; the notation of matrices is used. The major portion of the treatment is devoted to the vector equation $\gamma'(x)\cdot = \{A(x)\lambda + B(x)\}\gamma(x)\cdot$

and to the system comprised of this equation and linear boundary conditions. The elements of $A(x)$ and $B(x)$ are assumed to be differentiable, and λ is a complex parameter. The asymptotic form of the solutions of the equation, and the distribution of the characteristic values of the system under regular and particular irregular conditions are obtained if the roots $\gamma_j(x)$ $(j = 1, 2, \cdots, n)$ of the equation $|a_{ij}(x) - \delta_{ij}\gamma(x)| = 0$ are non-vanishing and distinct, and satisfy the condition $\arg\{\gamma_j(x) - \gamma_i(x)\} = c_{ij}$ (c_{ij} a constant). Finally the formal development of a vector of arbitrary functions, $F(x)\cdot$, into a series of characteristic functions is derived and is proved to converge when the points $\gamma_j(x)$ $(j = 1, 2, \cdots, n)$ are either constrained to lie on a line through the origin of the complex plane (in particular are all real), or form the vertices of a polygon which, as x varies, at most expands or contracts about the origin.*

This paper will appear in the PROCEEDINGS OF THE AMERICAN ACADEMY.

4. Mr. R. E. Langer: *Developments associated with a boundary problem not linear in the parameter.*

It appears that the ordinary differential equations which have heretofore been used as sources of characteristic functions in expansion problems are each equivalent to a system of linear, first order equations of the type

$$(1) \qquad g_i(x)u_i'(x) = L_i(u) + \lambda M_i(u) \quad (i = 1, 2, \cdots, n),$$

where $g_i \not\equiv 0$, while L_i and M_i are linear expressions in $u_1, \cdots, u_n$, $M_i \equiv 0$ for $i \neq n$, $M_n \not\equiv 0$.† The author discusses in this paper the expansion problem which results from choosing as a source of characteristic functions an equation which is equivalent to a system of type (1) in which $g_i \equiv 0$ for $i \neq n$, $g_n \not\equiv 0$, $M_i \not\equiv 0$ $(i = 1, 2, \cdots, n)$. The equation in question with boundary conditions is $P(\lambda)u'(x) = Q(\lambda, x)u(x)$, $u(a) = hu(b)$, $P(\lambda)$ and $Q(\lambda, x)$ being polynomials in λ which satisfy certain restrictions. It is shown that the characteristic values of the system cluster about certain points, and that n arbitrary functions may be formally developed with one determination of coefficients into series. These developments are shown to reduce under certain conditions to Four-

* TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), pp. 373–395.

† See a joint paper by Professor Birkhoff and the author, reported at this meeting, where a system in which $g_i \not\equiv 0$, $M_i \not\equiv 0$ $(i = 1, 2, \cdots, n)$, is treated.

ier's expansions and are proved in every case (with usual restrictions on $f_i(x)$) to converge like Fourier's series.

5. Professor L. P. Eisenhart: *Ricci's principal directions for a Riemann space and the Einstein theory.*

In 1904 Ricci developed the idea of principal directions in a Riemann space of n dimensions. In doing so he introduced the contracted curvature tensor, which is fundamental in the Einstein theory, and thus gave a geometrical interpretation to it. The principal directions are those for which the mean curvature takes maximum and minimum values, the mean curvature at a point for a direction h being the sum of the Riemann curvatures for the $n - 1$ directions determined by h and each of $n - 1$ directions orthogonal to h. The present paper points out that a space in which the principal directions are completely indeterminate may be thought of as possessing a homogeneous character. Applying these considerations to spaces of four dimensions, it is shown that the three types of space chosen by Einstein in 1914, 1917 and 1919 as spaces free from matter are of this homogeneous character, and include all types of such spaces.

6. Dr. Jesse Douglas: *Normal congruences and quadruply infinite families of curves.*

The present paper deals with a general geometric problem suggested by a theorem of Thomson-Tait and a converse theorem of E. Kasner: to classify quadruply infinite families of curves in space, $y'' = F(x, y, z, y', z')$, $z'' = G(x, y, z, y', z')$, with respect to normal congruences contained within them. The analysis is based on the equations of variation of the differential equations of the family, and leads to an infinite system of partial differential equations of the Monge-Ampère type. The infinite system of equations is discussed as to its complete integrability, resulting in the following classification of families of ∞^4 curves: (1) ∞^∞ normal congruences, where the infinitude is that of all the surfaces of space. This is the case of the theorems of Thomson-Tait and Kasner, where the family is of the "natural" type. (2) ∞^∞ normal congruences, where the infinite exponent represents the generality of two arbitrary functions of one argument. (3) ∞^∞ normal congruences, where the infinite exponent corresponds to one arbitrary function of one argument. (4) ∞^3 normal congruences. (5) ∞^2 normal congruences. (6) No normal congruences, or a finite number, or an infinite number (which must be ∞^1 or ∞^∞), but all confined to a triply infinite sub-family. This is the general case.

7. Dr. T. H. Gronwall: *Qualitative properties of the ballistic trajectory. Second paper.*

In the last section of the author's first paper with this title (ANNALS OF MATHEMATICS (2), vol. 21 (1920), pp. 44–65), it is shown that under certain very broad assumptions on the G- and H-functions, the velocity of the projectile has only a finite number of maxima and minima. The present paper gives further information regarding the existence and location of these extremes.

8. Dr. T. H. Gronwall: *The reflection of X-rays in a finite number of equidistant parallel planes.*

In the PHYSICAL REVIEW (May, 1921), Dr. Lamson has solved the corresponding, and simpler, problem for an infinite number of planes. The present paper reduces the problem for a finite number of planes to a non-linear difference equation of the second order, the solution of which is obtained algebraically.

9. Professor J. L. Coolidge: *The basis of mathematical probability.*

Certain historical and philosophic aspects of the mathematical foundations of the theory of probability were presented in this paper, which was read at the request of the programme committee.

10. Professor E. V. Huntington: *On the Alabama paradox in the problem of apportionment.*

If the problem of apportionment is regarded as a problem in minimizing some form of *total error*, there are 80 or more summation formulas to be considered. (See, for example, the writer's paper presented at the Wellesley meeting, September, 1921; this BULLETIN, vol. 28 (1922), p. 15.) Two of these formulas are known to lead to the method of major fractions, and two to the method of equal proportions. It is here shown that all the other known summation formulas lead to the *Alabama paradox*, that is, an increase in the total size of the House may involve a decrease in the representation of one of the states.

(On the general problem of apportionment, see papers in the QUARTERLY PUBLICATION OF THE AMERICAN STATISTICAL ASSOCIATION, September 1921, and December, 1921.)

11. Professor E. V. Huntington: *On the d'Hondt method of apportionment, and its counterpart.*

If the problem of apportionment is regarded as a problem in minimizing the inequality between every state and every other state, there are five different comparison formulas to be considered, leading to five distinct methods of apportionment. The "amount by which the population of the over-populated state must be reduced to bring it down to parity with the other state," taken in the sense of absolute differences, is shown to lead to the d'Hondt method. The "amount by which the population of the under-populated state must be increased to bring it up to parity with the other state," taken in the sense of absolute differences, is shown to lead to the contra-d'Hondt method, a new method which favors the small states extremely, just as the d'Hondt method favors the large states extremely. But each of these tests, if taken in the sense of relative differences, leads to the method of equal proportions.

12. Dr. G. A. Pfeiffer: *Theorems on sequences of sets of points.*

The first theorem of this paper is the following: If the sum of the sequence of sets of points, $\{A_n\}$, is compact in a metric space, then there exists a subsequence of $\{A_n\}$ whose lower closed limit* set is identical with its upper closed limit set and is between the upper and lower closed limit sets of the given sequence. An obvious modification of the definitions and proof of this theorem gives an analogous theorem for sets of point sets. Essential use is made of the above theorem in proving the following theorems: (1) If the sum of a sequence of connected sets of diameter $\leqq \delta$ is compact in a metric space, then the lower closed limit set of the sequence is contained in a closed connected set (a continuum) which has a diameter $\leqq \delta$ and which is contained in the upper closed limit set of the sequence. (2) If the sum of a sequence of continua $\{A_n\}$ is compact in a metric space and if A_n and A_{n+1} have a point in common and $\Sigma\delta_n$, where δ_n is the diameter of A_n, is convergent, then the closed limit set of $\{A_n\}$ contains one and only one point.

* Lower (upper) closed limit set = untere (obere) abgeschlossene Limes. See Hausdorff, *Grundzüge der Mengenlehre*, p. 236.

13. Professor C. A. Fischer: *The Fredholm theory of Stieltjes integral equations.*

A large part of the Fredholm theory has been developed for the equation $f(x) = \varphi(x) - \int_a^b \varphi(y) d_y K(x, y)$, by F. Riesz, where the integral defines a completely continuous transformation, without making use of the Fredholm determinant. In a previous paper Professor Fischer has found necessary and sufficient conditions which $K(x, y)$ must satisfy, in order that the transformation be completely continuous. In the present paper, modified definitions of Stieltjes integrals are given, and a set of conditions for $K(x, y)$ is found, under which the Fredholm determinant is defined for the above equation, and the classical Fredholm theory is applied directly. This theory neither includes, nor is included by, the Riesz theory.

14. Dr. J. L. Walsh: *A closed set of normal orthogonal functions.*

This paper presents a simple example of a closed normal orthogonal set of functions on the interval (0, 1). Each function takes on no value other than $+1$ or -1, except at a finite number of points; the nth function has precisely $n+1$ zeros (that is, sign-changes) in the interior of the interval; and a large class of arbitrary functions can be expanded in terms of these functions. The set is somewhat analogous to one considered by Haar, but has properties more closely allied to those of the more familiar sets of orthogonal functions.

15. Professor Arnold Emch: *Kinematics in a complex plane and some geometric applications.*

This paper appears in this number of this BULLETIN.

16. Professor J. F. Ritt: *On functions with integrals of elementary character.*

Let w be an integral of any algebraic function of z. It is a result of Liouville's that if w can be expressed in terms of elementary functions, then

$$(1) \qquad w = \alpha_0 + c_1 \log \alpha_1 + c_2 \log \alpha_2 + \cdots + c_n \log \alpha_n,$$

where each α is an algebraic function and each c a constant. The present paper gives an extension of Liouville's result. It is shown that if w is a solution of an elementary transcendental equation, then w is of the form (1). It follows that no elliptic function is a solution of an elementary transcendental equation. The same fact can be proved with respect to the integral logarithm and to many other transcendental functions.

17. Professor Eugenie M. Morenus: *Geometric properties of the system of all the curves of constant pressure in a plane field of force.*

The problem here considered is: In a plane field of force, to find smooth curves along which a constrained motion is possible such that the pressure of a moving particle against the curve shall remain constant. For one required constant pressure c there are ∞^3 such curves in the plane. The triply infinite system S_c has four geometrical properties which are shown to be sufficient to identify a set of ∞^3 curves as curves of constant pressure in a field of force. The quadruply infinite set of curves obtained by employing all possible values of c has eight geometrical properties which are sufficient to characterize completely a set of ∞^4 curves as curves of constant pressure. Eight additional properties of the quadruply infinite set are demonstrated.

18. Professor W. C. Graustein: *Spherical representation of conjugate systems and asymptotic lines.*

It is well known that a system of curves on the Gauss sphere represents a conjugate system of curves on each of infinitely many surfaces. It is here shown that, if the ratio of the radii of normal curvature in the conjugate directions is prescribed subject to a certain condition, one of the required surfaces is determined to within its homothetics. The general result is peculiarly adaptable to the important special cases of conjugate systems which have equal point or equal plane invariants or are isothermal-conjugate. It also lends itself readily to the treatment of lines of curvature. By similar methods applied to asymptotic lines a generalization of the theorem of Dini, giving the condition under which a system of curves on the sphere is the spherical representation of the asymptotic lines on a surface, is obtained. The paper will appear in the ANNALS OF MATHEMATICS.

19. Mr. Charles Manneback: *The distribution of current in a long cylindrical conductor.*

The problem of determining the distribution of density of an alternating current in a cylindrical conductor under the influence of a remote return parallel wire leads to the well known partial differential equations of Laplace and Poisson, with boundary conditions. The solution of the problem has not received hitherto a satisfactory answer. In the present

paper the equivalent integral equation

$$i(r,\theta) = \frac{\alpha^2 I}{2\pi}\log\frac{\delta}{D} + \frac{\alpha^2}{2\pi}\int\int i(\rho,\phi)\log\frac{d}{\rho}\,dS(\rho,\phi)$$

is set up and the following solution, which is a uniformly convergent series,*

$$i(r,\theta) = -\sum_{n=1}^{\infty}\frac{I}{\pi a^2}\left(\frac{a}{\delta}\right)^n j\alpha a\frac{J_n(j\alpha r)}{J_{n-1}(j\alpha a)}\cos n\theta,$$

is derived. Several related problems are also treated. The method appears capable of quite wide application.

20. Mr. A. Press: *Operational solution of equations of nth degree.*

An equation of the form $f(y) = a$ can be solved operationally if a mathematical operator f^{-1} can be found such that operating on both sides of the equation gives $f^{-1}\cdot f(y) = f^{-1}(a) = y$; f^{-1} is the inverse functional form of f. The operator multiplication (X) of ordinary algebra provides the basis of the required functional form with the understanding that $X^m\cdot y = f(y) = y^m$. The f in an equation of the nth degree then takes the form $f = a_mX^m + a_{m-1}X^{m-1} + \cdots = \Sigma a_mX^m$. The corresponding inverse of f is assumed of the form $f^{-1} = \Sigma a_pX^p$ which may be an ascending or descending series of implied multiplications or roots. By developing the corresponding algebra of the multiplication X as an operator either the inversion of series is accomplished or the form of resolution first obtained by McClintock. The operational developments in ascending form give the real roots and in descending form the complex roots. The algebra is particularly valuable in determining the complex roots of Heaviside's determinantal equations.

21. Professor T. R. Hollcroft: *Maximal cuspidal curves.*

A maximal cuspidal curve is an algebraic plane curve which has the greatest number of cusps possible for a given order n and genus $p \leqq \frac{1}{6}(n-2)(n-3)$. For $p > \frac{1}{6}(n-2)(n-3)$ all the double points may be cusps so there is no limit to the number of cusps except the genus. First, formulas are found

* r, θ and ρ, ϕ = polar coordinates of points of the conductor's cross-section; I = total return current; i = density of induced current in the conductor; a = radius of the conductor; δ = distance between centers of the parallel wires; D = distance between the return wire and point r, θ; d = distance between points r, θ and ρ, ϕ; dS = element of the conductor's cross-section area; α = a complex quantity of argument $\pi/4$.

giving the singularities of maximal cuspidal curves from genus 0 to p_0, the genus of the curve of lowest genus of the minimum class m_0, and the number of curves of class m_0 (which are always maximal cuspidal curves) and their singularities. Next, beginning with the self-dual maximal cuspidal curves, formulas are derived giving the number of maximal cuspidal curves for each class from n down to m_0. From these the maximum and minimum genus of each class is found, thence an inequality giving the class of any maximal cuspidal curve of given order and genus $p > p_0$, and from this an inequality giving the maximum number of cusps for any algebraic curve of given order and genus $p > p_0$. Using this inequality together with the one found for $p \leqq p_0$, we have the following: The maximum number of cusps for any algebraic plane curve of order n and genus $p \leqq \frac{1}{6}(n-2)(n-3)$ is given by the greatest integer k satisfying the smaller of these two inequalities when both are real or the first when the second is imaginary: (1) $k \leqq \frac{3}{2}(n-2) + 3p$; (2) $k \leqq 2(n+p) - \frac{1}{2}(11 + \sqrt{24p - 8n + 25})$.

22. Professor I. J. Schwatt: *Method for the separation into partial fractions of powers of trigonometric functions.*

The separation into partial fractions of $\tan^p x$, $\operatorname{ctn}^p x$, $\sec^p x$ and $\csc^p x$ depends on the evaluation of expressions like

$$\left(1 \pm \frac{2x}{(2n+1)\pi}\right)^p \tan^p x, \quad x = \mp\frac{2n+1}{2}\pi;$$

$$\left(1 \pm \frac{x}{n\pi}\right)^p \operatorname{ctn}^p x, \quad x = \mp n\pi; \text{ etc.}$$

The methods devised and the results obtained in this paper are believed to be new.

23. Professor I. J. Schwatt: *The expansion of the continued product* $\Pi(x+k)$.

The methods developed in the preceding paper have enabled the author to obtain the general term of the expansion of the continued product $\Pi(x+k)$.

R. G. RICHARDSON,
Secretary

A PROPERTY OF CONTINUITY*

BY D. C. GILLESPIE

If ξ and η are two points of the interval (a, b) in which the function $f(x)$ is continuous, then the function takes on all values between $f(\xi)$ and $f(\eta)$ as x changes from ξ to η. This property, which we shall designate by (A), is common to all continuous functions, but it is possessed also by other functions. It was shown, for example, by Darboux† that all derived functions possess property (A), and it was pointed out by Lebesgue‡ that still other functions which are neither continuous nor derived have the property.

The present note is concerned with functions having property (A). The character of the discontinuities that such a function may have is shown. Additional conditions which are sufficient to insure that the functions be continuous or continuous and monotone follow. A theorem stating that when all the discontinuities of a function are of a certain kind it has property (A) is proved, and a function having property (A) and having its set of points of continuity and its set of points of discontinuity each everywhere-dense is constructed.

If c is a point of discontinuity of a function $f(x)$ having property (A), then in any interval about c the function takes on all values between its maximum§ and minimum at c. If $f(x)$ is unbounded, for example from above, in the neighborhood of c, the function could have no maximum at c, in which case, obviously, $f(x)$ would have, in any interval about c, all values greater than its minimum at c.

As an immediate consequence of the character of the discontinuities of a function having property (A), it follows that *such a function will be continuous unless the set of values it assumes an infinite number of times fills at least one interval.* The converse of this statement is not true. An examination

* Presented to the Society, Sept. 7, 1920.

† ANNALES DE L'ECOLE NORMALE (2), vol. 4 (1875), pp. 109, 110.

‡ *Leçons sur l'Intégration*, p. 92.

§ Hobson, *Theory of Functions of a Real Variable*, p. 234, for definition of maximum and minimum of a function at a point.

of the Weierstrass non-differentiable continuous function reveals the fact that it takes on each value between its value at 0 and its value at 1 an infinite number of times for values of the variable in the interval (0, 1).

If then a function having property (A) takes on no value more than once, it is continuous and hence also monotone. This idea is capable of a slight extension and then comes to an abrupt end. Since a function having property (A) and taking on no value an infinite number of times is continuous, we shall state the extension for continuous functions.

If $f(x)$, continuous in (a, b), assumes no value more than twice, it is possible to divide the interval into three parts in each of which $f(x)$ is monotone.

It will be convenient to consider two cases:

(1) $f(a) = f(b)$,

(2) $f(a) \neq f(b)$.

In case (1), $f(x)$ must have either its greatest or its least value at a point ξ within (a, b). In each of the intervals (a, ξ) and (ξ, b) the function takes on every value between $f(a) = f(b)$ and $f(\xi)$ one and therefore only one time, hence in each of these intervals $f(x)$ is monotone.

In case (2) assume $f(a) < f(b)$. Suppose now $f(x)$ has its least value at a point ξ and its greatest value at a point η, where ξ and η are, of course, not necessarily distinct from a and b.* It follows from the assumption $f(a) < f(b)$ that $\xi < \eta$, for if η were less than ξ, the function would have the value $f(b)$, for example, once in the interval (a, η), once in the interval (η, ξ) and at b. In the interval (a, ξ) the function has its least value at ξ and it must have its greatest value at a, since, if the function were greater than or equal to $f(a)$ within (a, ξ), it would assume the value $f(a)$ at least three times: once in the interval (ξ, η), once within (a, ξ) and at a. This function having in the interval (a, ξ) its greatest value at a and its least value at ξ, can take on no value two times in (a, ξ). For suppose the function has the value M at two points p_1 and p_2 within (a, ξ), then, unless $f(x) = M$ has a

* If $f(x)$ have an extreme value at a or b, take $\xi = a$ or $\eta = b$.

root within (p_1, p_2), either $f(x) > M$ for all points within (p_1, p_2) and $f(x)$ has all values between M and its greatest value in (p_1, p_2) at least twice in the interval (p_1, p_2) and once in the interval (a, p_1); or $f(x) < M$ for all points within (p_1, p_2) and $f(x)$ has all values between M and its smallest value in (p_1, p_2) at least twice in (p_1, p_2) and once in (p_2, ξ). The function then has each value between $f(a)$ and $f(\xi)$ one and only one time in (a, ξ) and is therefore monotone. In the same way one shows that $f(x)$ is monotone in (ξ, η) and (η, b), the extreme values of the function for the intervals occurring at the end-points of the intervals.

If we assume that $f(x)$ takes on no value more than three times and that $f(a) = f(b)$, a similar argument will show that the interval may be divided into three parts in each of which the function is monotone.

This idea is capable of no further extension, since a continuous function may take on no value more than three times in an interval and yet be such that it is impossible to divide the interval into a finite number of parts in which the function is monotone. This statement is verified by the example:

$f(x) = \pi x + x^2 \sin(\pi/x)$ for $0 < x \leqq 1$,

$f(0) = 0$.

The function has proper maximum points at $1/2, 1/4, 1/6, \cdots$ and proper minimum points at the roots of the equation $\tan(\pi/2x) = -(2x/\pi)$, which occur in the intervals $(1/2, 1)$, $(1/4, 1/3)$, $(1/6, 1/5)$, etc. In the interval $[1/(2n+1), 1/(2n-1)]$ the function has one maximum point and one minimum point, hence takes on no value more than three times in this interval. Moreover,

$$\frac{\pi}{2n+1} \leqq f(x) \leqq \frac{\pi}{2n-1}$$

for

$$\frac{1}{2n+1} \leqq x \leqq \frac{1}{2n-1},$$

$$f(x) > \frac{\pi}{2n-1} \quad \text{for} \quad x > \frac{1}{2n-1}$$

and

$$f(x) < \frac{\pi}{2n+1} \quad \text{for} \quad x < \frac{1}{2n+1}.$$

We have shown that a function having property (A) assumes in any interval about a point of discontinuity every value between its maximum and minimum at this point. The converse of this is not true; for example, $f(x) = 1 + x + \sin(\pi/x)$ for $0 < x \leqq 1$, $f(0) = 1$ and $f(x) = x$ for $-1 \leqq x < 0$ has at zero its only discontinuity. In every interval about zero it assumes every value between 0, its minimum at 0, and 2, its maximum at 0, and although $f(-1) = -1$ and $f(1) = 2$, $f(x) = 0$ has no root in $(-1, 1)$. It is to be observed that the values between 0 and 2 are not assumed by the function *in any interval whose upper end-point is* 0.

THEOREM. *If a function $f(x)$ at every point of discontinuity c takes on all values between its maximum and minimum at c in every interval of which c is an end-point, then the function has property (A).*

Suppose ξ_1 and η_1, $\xi_1 < \eta_1$, are two points of the interval (a, b) in which $f(x)$ is defined, and that $f(\xi_1) < M < f(\eta_1)$; we shall show that $f(x) = M$ has a root between ξ_1 and η_1. Consider the set of points in (ξ_1, η_1) at which $f(x) > M$ and take the lower limit p_1 of this set. If $p_1 = \xi_1$, then, since ξ_1 is the lower limit of a set of points at which $f(x) > M$, the maximum of $f(x)$ at ξ_1 is not less than M. If this maximum of $f(x)$ at ξ_1 is greater than M, then $f(x) = M$ has a root in every interval of which ξ_1 is an end-point. If $p_1 > \xi_1$, then there is no point within (ξ_1, p_1) for which $f(x) > M$, hence the maximum of $f(x)$ at p_1 is M. In any case, whether p_1 is equal to ξ_1 or greater than ξ_1, the equation $f(x) = M$ either has a root in (ξ_1, η_1) or $f(p_1) < M$ and the maximum of $f(x)$ at p_1 is M. Suppose $M - f(p_1) = \epsilon$; now choose a positive $\delta < [(\eta_1 - p_1)/2]$ and such that $f(x) < M + \epsilon/2$ in the interval $(p_1, p_1 + \delta)$. Within this interval there is a point η_2 at which $f(\eta_2) > M$. Now it may be established likewise that either $f(x) = M$ has a root in (p_1, η_2) or there exists at the

upper end of (p_1, η_2) a subinterval (ξ_2, η_2) such that $f(\xi_2) < M$ and $f(x) > M - (\epsilon/2)$. Thus in the interval (ξ_2, η_2)

$$|f(x) - M| < \epsilon/2.$$

Continuing this process, we obtain a sequence of intervals (ξ_n, η_n) each lying within the preceding one and of length less than $(\eta_1 - \xi_1)/2^{n-1}$ for which $f(\xi_n) < M$, $f(\eta_n) > M$, $|f(x) - M| < \epsilon/2^{n-1}$ for x in (ξ_n, η_n). Hence at the one point p common to this set of intervals $f(p) = M$.

It will require only obvious changes in the preceding argument to prove a more general theorem*. Using the four symbols $\bar{f}(c+0)$, $\underline{f}(c+0)$, $\bar{f}(c-0)$ and $\underline{f}(c-0)$ to designate the upper and lower limits† of $f(x)$ to the right and left of c respectively, this theorem may be stated:

If at every point of discontinuity c of $f(x)$, $\underline{f}(c+o) \leqq f(c) \leqq \bar{f}(c+0)$, $\underline{f}(c-0) \leqq f(c) \leqq \bar{f}(c-0)$ and $f(x)$ takes on every value between $\underline{f}(c+0)$ and $\bar{f}(c+0)$ in every interval whose left end-point is c and every value between $\underline{f}(c-0)$ and $\bar{f}(c-0)$ in every interval whose right end-point is c, then $f(x)$ has property (A).

The conditions thus imposed upon $f(x)$ are not only sufficient but also clearly necessary for the existence of property (A).

As illustrations of functions having property (A) we have on the one hand continuous function or certain functions with a finite number of discontinuities, and on the other hand totally discontinuous functions.‡

The following function § defined by a uniformly convergent series has property (A) and has its set of points of continuity and its set of points of discontinuity each everywhere-dense: ‖

$$f(x)\| = \sum_{n=1}^{\infty} \frac{1}{n^2} g(x - r_n),$$

* E. R. Hedrick pointed out to me that the argument really proved this more general theorem.

† Hobson, loc. cit., p. 231.

‡ Lebesgue, loc. cit., p. 90.

§ Pierpont, *Theory of Functions of a Real Variable*, Vol. II, p. 463, Ex. 1.

‖ W. A. Hurwitz suggested the probability of the function defined possessing the properties enumerated. The present note grew out of a study of this example.

where $r_1, r_2, r_3, \cdots$, denotes the set of rational numbers, $g(x) = \sin(\pi/x)$ for $x = 0$ and $g(0) = 0$.

The function $f(x)$ is continuous at all irrational points and has a discontinuity of the second kind with oscillation equal to $2/n^2$ at each rational point r_n. To prove the function has property (A), it is sufficient, in consequence of the theorem just established, to show that $f(x) = M$, where $f(r_n) - (1/n^2) < M < f(r_n) + 1/n^2$, has a root in every interval of which r_n is an end-point. From the definition of $f(x)$ it is obvious that corresponding to any positive number ϵ there exists a positive number δ such that the oscillation of $f(x)$ at any point lying inside the interval $(r_n, r_n + \delta)$ is less than ϵ. Within $(r_n, r_n + \delta)$ there is a point η_1 at which $f(\eta_1) > M$ and then within (r_n, η_1) a point ξ_1 at which $f(\xi_1) < M$. By taking the upper limit of those points in (ξ_1, η_1) at which $f(x) < M$ and proceeding as before, a convergent sequence of intervals (ξ_m, η_m) where $f(\xi_m) < M < f(\eta_m)$ and for which the oscillation at each point is less than $\epsilon/2^m$ may be constructed. At the one point p common to the set of intervals, $f(p) = M$. The same is true, of course, of intervals bounded above by r_n.

One could infer, from facts already established, the existence of a function having property (A) and having its set of points of continuity and its set of points of discontinuity each everywhere-dense. For functions which are everywhere oscillating and which nevertheless have a derivative at every point of an interval have been constructed.* The derived function has, of course, property (A); this derived function being the limit of a sequence of continuous functions has its points of continuity everywhere-dense;† its points of discontinuity are also everywhere-dense since it is both positive and negative in every interval.

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* Köpcke, MATHEMATISCHE ANNALEN, vols. 34 and 35.

† Baire, *Leçons sur les Fonctions Discontinues*, p. 98.

KINEMATICS IN A COMPLEX PLANE AND SOME GEOMETRIC APPLICATIONS *

BY ARNOLD EMCH

1. *Introduction.* From an elementary standpoint one is apt to consider geometry in a complex plane as an accessory of heuristic value for function theory. In a more advanced sense one recognizes the fundamental importance and intrinsic value of the geometric problem of partition of the complex plane by circular arcs in connection with the properties and classification of certain linear substitution groups, with the corresponding automorphic functions, and, in particular, with the theory of algebraic curves and their Riemann surfaces and uniformization. But even in a more elementary sense, the complex plane is the natural medium for the solution of certain specific geometric problems. As an example may be mentioned the "geometry of the polynomial," involving the theory of stelloid and circular curves and their focal properties.†

Also a number of problems in geometric kinematics may be solved conveniently in a complex plane, as has been shown by Koenigs,‡ Study,§ and others. In the present paper I shall show by further examples of this kind the simplicity and elegance of the complex treatment.

2. *Similar Triangles.* As can easily be verified, a necessary and sufficient condition for the equi-sensed similitude of two triangles $z_1z_2z_3$, $z_1'z_2'z_3'$ is the vanishing of the determinant

$$(1) \qquad \begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} \equiv 0.$$

* Presented to the Society, Feb. 25, 1922.

† See Emch, *On a certain generation of rational circular and isotropic curves*, this BULLETIN, vol. 25, pp. 397–404 (1919), and also *On plane algebraic curves with a given system of foci*, same volume, pp. 157–161.

‡ *Leçons de Cinématique: Les imaginaires dans la cinématique du plan*, pp. 324–332 (1897).

§ *Vorlesungen über ausgewählte Gegenstände der Geometrie.* Erstes Heft: *Ebene analytische Kurven und zu ihnen gehörige Abbildungen*, pp. 1–18 (1911).

Denoting by $\bar{z}$ the conjugate of z, two triangles $z_1z_2z_3$, $z_1'z_2'z_3'$ are similar with the sense inverted when

$$\begin{vmatrix} \bar{z}_1 & z_1' & 1 \\ \bar{z}_2 & z_2' & 1 \\ \bar{z}_3 & z_3' & 1 \end{vmatrix} \equiv 0. \tag{2}$$

For an equilateral triangle $z_1z_2z_3$ we simply have to set up the condition that the triangle $z_2z_3z_1$ in the same order be similar to the triangle $z_1z_2z_3$; i.e.,

$$\begin{vmatrix} z_1 & z_2 & 1 \\ z_2 & z_3 & 1 \\ z_3 & z_1 & 1 \end{vmatrix} \equiv 0.$$

This reduces to

$$z_1^2 + z_2^2 + z_3^2 - z_2z_3 - z_3z_1 - z_1z_2 = 0 \tag{3}$$

as a necessary and sufficient condition that the triangle $z_1z_2z_3$ be equilateral.

3. *The Group of Movements in a Plane.* The linear transformation

$$z' = az + b \tag{4}$$

may be considered as a movement (including uniform dilatation) of the z-plane into a new position indicated by z' and referred to the same system of coordinates (original z-plane). Putting $a = r_1e^{i\alpha_1}$, $b = r_2e^{i\alpha_2}$, $z = re^{i\theta}$, (4) becomes

$$z' = r_1re^{i(\theta+\alpha_1)} + r_2e^{i\alpha_2}, \tag{5}$$

and it is seen at once that the movement is equivalent to the effect of the succession of substitutions

(*a*) S_1(*similitude*): $z_1 = r_1 \cdot z$;

(*b*) S_2(*rotation*): $z_2 = z_1 \cdot e^{i\alpha_1}$;

(*c*) S_3(*translation*): $z_3 = z' = z_2 + r_2e^{i\alpha_2}$.

The totality of all movements (4) forms a continuous projective four-parameter group and contains (*a*), (*b*), (*c*) as subgroups. The invariant points in the movement from the z- to the z'-plane are $z = b/(1 - a)$ and $z = \infty$.

When z describes a figure in the original plane, z' describes a similar figure in the displaced plane with the coefficient of dilatation equal to r_1. In particular, when z describes either a straight line or a circle, z' will describe a straight line or a

circle. Moreover, since

$$dz' = a \cdot dz, \qquad \text{and} \qquad |dz'| = r_1 \cdot |dz|,$$

the ratio of the velocities of z and z' is constant. Thus when z describes a straight line with uniform velocity, z' will describe a straight line with uniform velocity. When z describes a circle, z' describes a circle with the same angular velocity, i.e., z and z' describe their respective circles in the same time.

4. *On the Movement of a Triangle in Deformation and Remaining Similar to Itself.* When z describes the unit-circle $z = e^{i\theta}$,

$$z' = r_1 e^{i(\theta + a_1)} + r_2 e^{ia_2} \tag{6}$$

describes obviously a circle with $r_2 e^{ia_2}$ as a center and r_1 as a radius. Consider now the triangle formed by the origin, the unit-point on the real axis and the fixed point $\rho e^{i\beta}$. On the line zz' which connects the point z on the unit-circle to the point z', corresponding to z by (4), erect a triangle $z_3 z z'$ similar, in the same order, to the triangle $\rho e^{i\beta}$, 0, 1. Then

$$\begin{vmatrix} z_3 & \rho e^{i\beta} & 1 \\ e^{i\theta} & 0 & 1 \\ r_1 e^{i(\theta + a_1)} + r_2 e^{ia_2} & 1 & 1 \end{vmatrix} = 0.$$

From this

$$-z_3 + e^{i\theta} - \rho e^{i(\theta+\beta)} + \rho r_1 e^{i(\theta + a_1 + \beta)} + \rho r_2 e^{i(a_2+\beta)} = 0;$$

$$z_3 = \{\rho r_1 e^{i(a_1+\beta)} - \rho e^{i\beta} + 1\} e^{i\theta} + \rho r_2 e^{i(a_2+\beta)}. \tag{7}$$

As (7) has the same form as (6) and θ is the only variable, z_3 clearly describes a circle with $|\rho r_1 e^{i(a_1+\beta)} - \rho e^{i\beta} + 1|$ as a radius and $\rho r_2 e^{i(a_2+\beta)}$ as a center. Obviously, z_3, z', z describe the circles with the same angular velocities, i.e., they simultaneously describe the circles completely. Denote the centers of the circles described by z, z', z_3 by c_1, c_2, c_3, so that $c_1 = 0$, $c_2 = r_2 e^{ia_2}$, $c_3 = \rho r_2 e^{i(a_2+\beta)}$. Comparing this triangle with the original fixed triangle 0, 1, $\rho e^{i\beta}$, we see that

$$\begin{vmatrix} 0 & 0 & 1 \\ r_2 e^{ia_2} & 1 & 1 \\ \rho r_2 e^{i(a_2+\beta)} & \rho e^{i\beta} & 1 \end{vmatrix} \equiv 0.$$

From this it follows that the two triangles are similar.

Consider next a point z_4 which remains always similarly attached to the triangle $zz'z_3$, so that also $zz'z_4$ remains similar

to itself. But such a triangle $zz'z_4$ arises from a definite fixed similar triangle 0, 1, $\delta e^{i\nu}$, just as $zz'z_3$ arises from 0, 1, $\rho e^{i\beta}$. Hence also z_4 describes a circle, and $\triangle c_1c_2c_4 \sim \triangle zz'z_4$, i.e., if a fourth point z_4 remains similarly attached to the triangle zz_1z_3, then z_4 describes a circle whose center c_4 is similarly attached to $c_1c_2c_3$. The same situation prevails for other similarly attached points $z_5, z_6, \cdots$, and as the geometric result does not depend upon the orientation of the complex plane, we may state the following theorem.

THEOREM 1. *If a variable closed or unclosed polygon*

$$(z) = z_0z_1z_2 \cdots z_n$$

of $(n+1)$ *vertices remains similar to some arbitrary fixed polygon*

$$(a) = a_0a_1a_2 \cdots a_n,$$

and any two of its vertices describe two fixed circles with the same angular velocities, then all other vertices describe circles with the same angular velocities. The polygon

$$(c) = c_0c_1c_2c_3 \cdots c_n$$

of the centers of these circles is similar to the fixed polygon (a).

From this follows obviously the corollary:

COROLLARY.* *If any two vertices of a regular variable polygon* $(z) = z_1z_2 \cdots z_n$ *of* n *vertices and center* z_0 *describe two circles with the same angular velocities, then the remaining* $n-2$ *vertices describe circles with the same angular velocities. The centers* $c_1c_2c_3 \cdots c_n$ *of these circles form a fixed regular polygon* (c), *whose center* c_0 *is the center of the circle described by the center* z_0 *of the variable polygon* (z).

5. *Case of Movements on Straight Lines.* It is not difficult to show that the theorem of the preceding section holds, when two points of the polygon (z) describe straight lines with uniform velocities. We may restate the result.

THEOREM 2. *If any two points of a variable polygon* (z), *which remains similar to some fixed polygon* (a), *describe two straight lines with uniform velocities, then all other points of* (z) *describe straight lines with uniform velocities.*

* This contains Study's theorem on the square, loc. cit., p. 16, as a particular case. It is obviously not necessary to make a restriction in the choice of the two vertices as Study does.

In particular, when (z) is a regular polygon, its center z_0 also describes a straight line.

We propose next to discover whether it is possible to find triangles with their vertices on three arbitrarily given fixed lines and similar to a fixed triangle 0, 1, $re^{i\alpha}$. Let $\lambda_1, \lambda_2, \lambda_3$ be three variable parameters; then the three lines may be written in the form

$$z_1 = a_1 + \lambda_1 e^{i\alpha_1},$$
$$z_2 = a_2 + \lambda_2 e^{i\alpha_2},$$
$$z_3 = a_3 + \lambda_3 e^{i\alpha_3}.$$

For every set of values $\lambda_1, \lambda_2, \lambda_3$ the three points z_1, z_2, z_3 form a triangle on the three lines. Similitude to the fixed triangle requires

$$\begin{vmatrix} a_1 + \lambda_1 e^{i\alpha_1} & 0 & 1 \\ a_2 + \lambda_2 e^{i\alpha_2} & 1 & 1 \\ a_3 + \lambda_3 e^{i\alpha_3} & re^{i\alpha} & 1 \end{vmatrix} = 0.$$

Expanding this and separating real from imaginary, we get an expression of the type

$$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 + A_4 + i(B_1\lambda_1 + B_2\lambda_2 + B_3\lambda_3 + B_4) = 0$$

in which the A's and B's and λ's are real. In order that this be satisfied for real values of the λ's, there must be

$$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 + A_4 = 0,$$
$$B_1\lambda_1 + B_2\lambda_2 + B_3\lambda_3 + B_4 = 0,$$

which shows the *existence of a simply infinite set* of solutions $(\lambda_1, \lambda_2, \lambda_3)$, or of such triangles. Hence we have the following theorem.

THEOREM 3. *If three lines l_1, l_2, l_3 are given, it is always possible to describe them simultaneously by three points of a one-parameter variable triangle which remains similar to a fixed triangle. The vertices of the triangle describe the sides $l_1l_2l_3$ with definite uniform velocities.*

Now choose a fourth point z_4 so that the variable quadrangle $z_1z_2z_3z_4$, with z_1, z_2, z_3 moving on $l_1l_2l_3$, remains similar to a fixed quadrangle $a_1a_2a_3a_4$; then z_4 describes a straight line l_4.

If four lines $l_0l_1l_2l_3$ are given, and we let z_1, z_2, z_3 describe l_1, l_2, l_3 as before, then z_4 describes a line l_4. When z_4, moving on l_4, reaches the intersection z_0' of l_4 with l_0, z_1, z_2, z_3 will have reached certain positions $z_1'z_2'z_3'$ on l_1, l_2, l_3, respectively, so that $z_0'z_1'z_2'z_3' \sim a_1a_2a_3a_4$.

THEOREM 4. *It is always possible to inscribe in a given quadrilateral a quadrangle similar to a given quadrangle. In particular, it is always possible to inscribe a square in a given quadrangle.**

Theorem I is also true when one of the two circles determining the movement degenerates into a point circle, or when one degenerates into a point circle and the other into a straight line. Thus follows

THEOREM 5. *If a variable triangle $z_1z_2z_3$ remains similar to a fixed triangle, and z_3 is stationary, while z_1 describes a straight line with uniform velocity, then z_2 describes another straight line with uniform velocity.*

As a particular case assume an isosceles right triangle $z_1z_2z_3$ with $z_3 = ei$ fixed on the axis of imaginaries and $z_1 = \lambda$, the vertex of the right angle, describing the axis of reals. Then $z_2 - \lambda = -(ei - \lambda)i$, and $z_2 = e + \lambda(1 + i)$, i.e., z_2 describes a line through the point e on the real axis and making an angle of 45° with the positive part of the real axis. This result leads to the following method.

6. *Inscribing a Square in a Quadrilateral.* Let $l_1l_2l_3l_4$ be the sides of a proper quadrilateral following each other in the positive sense. On l_1 choose a trial point A_1 and drop a perpendicular A_1Q_1 on l_2. From Q_1 measure off on l_2 in the positive sense $Q_1A_2' = A_1Q_1$. Through A_2' draw a line g_1 making an angle of 45° with the positive direction of l_2, and intersecting l_3 in A_3. Then A_1A_3 form two opposite vertices of a square on $l_1l_2l_3$ and we can easily find the other vertices A_2, A_4. Repeat the same construction, starting with a second

* The second part of this theorem is well known. See Dr. Hebbert's papers, ANNALS OF MATHEMATICS (2), vol. 16, pp. 38–42; 61–71 (1914–15), and references given there.

trial point B_1 on l_1, resulting in a second square $B_1B_2B_3B_4$ with $B_1B_2B_3$ on $l_1l_2l_3$. Now when a variable square moves with three vertices on three lines $l_1l_2l_3$, the fourth vertex describes a line l_4'. $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two positions of this square, and A_4 and B_4 determine the line l_4'. Hence, the point of intersection P_4 of l_4' and l_4 is a vertex of the inscribed square. Repeating the above construction in the reversed order, starting with P_4 on l_4, easily gives the square $P_4P_3P_2P_1$ inscribed in $l_1l_2l_3l_4$.

This however is not the only square. A second square is obtained by measuring off $Q_1A_2^* = -Q_1A_2$ in the negative sense, by drawing g_1^* through A_2^*, making an angle of 45° with the negative direction of l_2, and proceeding with the construction as explained above. Starting again with l_1 and choosing the order $l_1l_2l_4l_3$, there are again two squares whose vertices lie in succession on $l_1l_2l_4l_3$. Likewise there are two squares for the order $l_1l_3l_2l_4$. The remaining three orders $l_1l_4l_3l_2$, $l_1l_3l_4l_2$, $l_1l_4l_2l_3$ are cyclic substitutions of the above three and do not produce any new squares.

THEOREM 6. *There are, in general, six squares that may be inscribed in a proper quadrilateral. For each of the three non-cyclic orders there are two squares of opposite sense.*

When there exist two inscribed squares with the same sense for a given order, as for $l_1l_2l_3l_4$, then l_4' coincides with l_4, and there exists an infinite number of solutions.

THEOREM 7. *When for a given order of a quadrilateral there exist two inscribed squares with the same sense, then there exist an infinite number of inscribed squares for the same order.*

Thus a problem of old standing is solved in a complete* and very simple manner by preparing the way in the complex field.

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* Carnot in his trigonometric solution of the problem (*Géométrie de Position*, p. 374) stated the possibility of three solutions only.

NOTE ON SOME RESULTS CONCERNING FERMAT'S LAST THEOREM

BY H. S. VANDIVER

Let us suppose that the equation

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in integers prime to each other and that p is an odd prime. We shall divide the discussion into two parts. We shall refer to the case where xyz is prime to p as Case I, and to the case where $xyz \equiv 0 \pmod{p}$ as Case II. The present note will be confined to statements of new results obtained by the writer for both cases, without proofs.

1. *Case* I. *Transformations of the Kummer Criteria.* If

$$f_a(t) = \sum_{r=1}^{p-1} r^{a-1} t^r,$$

then it is known that, if (1) is satisfied in Case I, we have

$$\begin{cases} f_{p-1}(t) \equiv 0 \pmod{p}, \\ B_n f_{p-2n}(t) \equiv 0 \pmod{p}, \end{cases} \tag{2}$$

$n = 1, 2, \cdots, \mu - 1$, and $-t \equiv x/y, y/x, z/x, x/z, y/z, z/y, \pmod{p}$, $\mu = (p-1)/2$, where the B's are the Bernoulli numbers, $B_1 = 1/6$, $B_2 = 1/30$, etc. These congruences are known as the Kummer criteria. Transformations of them have yielded the following conditions for Case I:

$$\Big(\sum_{r=0}^{p-1} (2r+1)^i t^r\Big)\Big(\sum_{r=0}^{p-1} (2r+1)^{p-2-i} t^r\Big) \equiv 0 \pmod{p},$$

$$(i = 1, 2, \cdots, \mu - 1),$$

$$f_p(t_1) \equiv 0 \pmod{p^2},$$

$$-t_1 \equiv x/y, y/x, x/z, z/x, y/z, z/y \pmod{p^2}.$$

If e is the smallest integer such that $B_e \equiv 0 \pmod{p}$, we have

$$B_c f_{2k-1}(-t) \equiv 0 \pmod{p}, \tag{3}$$

$$(c = (p - 2k + 1)/2, \quad k = 2, 3, \cdots, e - 1).$$

Kummer and Mirimanoff* have shown that if (1) is satisfied in Case I, then

$$B_{\mu-1} \equiv B_{\mu-2} \equiv B_{\mu-3} \equiv B_{\mu-4} \equiv 0 \pmod{p}.$$

The relation (3) gives material for extension of these criteria.

* JOURNAL FÜR MATHEMATIK, vol. 128 (1905), pp. 45–68.

2. *Results from a New Method.* In efforts to supplement the relations (2) by independent conditions in Case I, the writer, by the use of new developments in the theory of cyclotomic fields, has derived the following criteria* for Case I:

$$B_s \equiv 0 \pmod{p^2}, \tag{4}$$

$$s = (hp+1)/2, \qquad h = p-4,\ p-6,\ p-8,\ p-10.$$

If $e = a^f$, where a is any prime and f any integer, then

$$\frac{(t^e-1)^p - t^{ep} + 1}{p} \equiv 0 \pmod{p}. \tag{5}$$

This relation possibly also holds if e is any integer, but the formulas which it seems necessary to examine in the general case are very complicated. The condition (5) gives the result: *If t belongs to the exponent n, modulo p, and n is the power of a prime, then*

$$1 + \frac{1}{2} + \cdots + \frac{1}{jv} \equiv 0 \pmod{p},$$

$$(v = (p-1)/n, \qquad j = 1, 2, \cdots, n-1).$$

Hence

$$q(n) \equiv \frac{n^{p-1} - 1}{p} \equiv 0 \pmod{p}.$$

In addition, these methods have yielded a new derivation of the known criteria $q(2) \equiv q(3) \equiv 0 \pmod{p}$ without use of (2) or of the criterion of Furtwängler.†

3. *Criteria derived by a second method.* Using transformations of some known theorems regarding cyclotomic fields, the following conditions have been found for Case I.

Let the principal ideal $(\omega(\alpha))$ be the pth power of any ideal in the field defined by $\alpha = e^{2i\pi/p}$ which is prime to z and p; then

$$f_{p-n}(s)\left[\frac{d^n \log \omega(e^v)}{dv^n}\right]_{v=0} \equiv 0 \pmod{p},$$

where $s \equiv -x/y \pmod{p}$, $n = 1, 2, \cdots, p-2$, and e is the Napierian base.

If we have two solutions of (1), say

$$x_1^p + y_1^p + z_1^p = 0,$$

$$x_2^p + y_2^p + z_2^p = 0,$$

* The first was previously stated in this BULLETIN, vol. 24 (1918), p. 472.

† WIENER BERICHTE, vol. 121 (1912), p. 589.

where z_1 is prime to z_2, then we have as a special case of the above,

$$f_{p-n}(S_1)f_n(S_2) \equiv 0 \pmod{p},$$

$S_1 \equiv -x_1/y_1$, $S_2 \equiv -x_2/y_2 \pmod{p}$; $(n = 1, 2, \cdots, p-1)$, which in turn includes the relations

$$f_{p-n}(t)f_n(1-t) \equiv 0 \pmod{p};$$

and we obtain also, by a modification of the theorem,

$$f_{p-n}(t)f_n(t) \equiv 0 \pmod{p},$$

n ranging as before, the latter being a known transformation of (2).

4. *Case* II. Under the following assumptions:

1. *None of the Bernoulli numbers*

$$B_k, \quad k = (sp+1)/2, \qquad (s = 1, 3, \cdots, p-4),$$

are divisible by p^2,

2. *The second factor of the class number of the field* $\Omega(\alpha)$ *is prime to* p,

I have proved that (1) is impossible in Case II. A special case of this gives the criterion that if (1) is satisfied in Case II, then the class number of $\Omega(\alpha)$ is divisible by p^2, which in turn may be shown to include the results which Kummer essayed to prove in his 1857 memoir on the last theorem. Taking into account the criteria (4) for Case I, we may cover both cases by the following statement.

Under assumptions 1 *and* 2, *the relation* (1) *is not satisfied in integers* x, y, z, *none zero, for a prime* $p > 2$.

It will be noted that this criterion may be tested for any prime in a finite number of steps.

Kummer* computed the value of the first factor of th eclass number of $\Omega(\alpha)$ for all primes less than 167. Assuming these computations correct, it may be shown that assumption 1 holds for all these primes. According to Kummer† the second assumption holds for all primes less than 100, and it has not been tested for larger primes.

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* BERLINER MONATSBERICHTE, 1874.

† BERLINER ABHANDLUNGEN, 1857, p. 73, verification of second assumption.

CONVEX DISTRIBUTION OF THE ZEROS OF STURM-LIOUVILLE FUNCTIONS*

BY EINAR HILLE

Introduction. In an investigation concerning the distribution in the complex plane of the zeros of the functions of the parabolic cylinder the author of the present note was confronted with the following problem. Given a linear differential equation of the second order of the type

$$w'' + G(z)w = 0, \tag{1}$$

where $G(z)$ is analytic in the region under consideration, *how does the nature of the argument of* $G(z)$ *affect the distribution of the zeros of a given solution* $w(z)$ *of* (1)?

We have some results which throw light on this general question. Roughly speaking, one may say that the argument of $G(z)$ affects the *orientation* of the zeros in the complex plane, whereas the absolute value of $G(z)$ seems to affect the *density* of the distribution. As the problem and the results obtained seem to have some intrinsic value aside from their bearing on the special problem, mentioned above, their separate publication may perhaps be justified.

1. *Bounded Argument of* $G(z)$. Let us consider a convex region B in the z-plane in which $G(z)$ is single-valued and analytic and in which, furthermore, the argument of $G(z)$ is restricted as follows:

$$\vartheta_1 \geqq \arg G(z) \geqq \vartheta_2 \quad (\pi > \vartheta_1 - \vartheta_2 \geqq 0). \tag{2}$$

Let us construct a set of polygonal lines (p) in B in the following manner. Take any point P_0 in the interior of B. Draw from this point all rays which form an angle θ_{P_0} with the positive real axis such that

$$-\tfrac{1}{2}\vartheta_1 < \theta_{P_0} < -\tfrac{1}{2}\vartheta_2. \tag{3}$$

Take a second point P_1 in B on any of these rays and use it as vertex for a second set of rays with the same limitation on the slope-angle θ_{P_1} as on θ_{P_0}. Then choose a third point P_2 and so on. The polygonal line so obtained forms an ele-

* Presented to the Society, Dec. 29, 1921.

ment p of our set (p). Repeat this construction for all interior points P_0 in B and for every possible choice of points $P_1, P_2, \cdots, P_n, \cdots$ on the resulting rays. The set of all these polygonal lines is the set (p). By the construction we define a positive direction on the line p, namely from P_n to P_{n+1} for all values of n; we always assume the line to be traced in this direction. Now take a solution $w(z)$ of (1) and put $W(z) = w(z)(dw/dz)$. Mark the zeros of $W(z)$ in B. Then we have the following theorem.

THEOREM I. *There is at least one polygonal line p_0 in the set (p), defined above, which joins all the zeros of $W(z)$ in B. The line p_0 is unique when and only when $G(z) = const.$*

The theorem is equivalent to the following one.

THEOREM II. *If a and b are two zeros of $W(z)$ in B, then*

$$-\tfrac{1}{2}\vartheta_1 + k\pi \leqq \arg(a-b) \leqq -\tfrac{1}{2}\vartheta_2 + k\pi,$$

where k is 0 or 1.

We have proved theorem II in another place* but the proof will be repeated for the benefit of the reader. Multiply equation (1) by $\overline{w}$, the conjugate of w, and integrate between a and b. After an integration by parts we obtain

$$(4) \qquad \left[\overline{w}\frac{dw}{dz}\right]_a^b - \int_a^b \left|\frac{dw}{dz}\right|^2 \overline{dz} + \int_a^b |w|^2 G(z)dz = 0.$$

But $W(z) = 0$ for $z = a$ and b, hence the integrated part disappears. Putting $z = a + re^{i\theta}$ where $\theta = \arg(b-a)$, we get

$$(5) \qquad e^{2i\theta}\int_0^{r_1} |w|^2 G(z)dr = \int_0^{r_1} \left|\frac{dw}{dz}\right|^2 dr, \quad (r_1 = |b-a|).$$

Thus, denoting the integral on the left-hand side in (5) by $\mathfrak{I}(a, b)$ we observe that $e^{2i\theta}\mathfrak{I}(a, b)$ is a real positive number.

Hence

$$(6) \qquad 2\theta = 2k\pi - \arg \mathfrak{I}(a, b).$$

But by (2) we conclude that $\vartheta_1 \geqq \arg \mathfrak{I}(a, b) \geqq \vartheta_2$, so that

$$(7) \qquad k\pi - \tfrac{1}{2}\vartheta_1 \leqq \theta \geqq k\pi - \tfrac{1}{2}\vartheta_2,$$

where the signs of equality hold only when $\vartheta_1 = \vartheta_2$, which clearly corresponds to $G(z) = \text{const.}$, when all the zeros lie on the same straight line.

* *On the zeros of Sturm-Liouville functions*, theorem XI, ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 16, nr. 17, 1921.

2. *Monotonic Variation of* arg $G(z)$. The line p_0 is not unique but the sub-set (p_0) of lines in (p) which pass through all the zeros of $W(z)$ in B meet these points in one and the same order. Suppose we enumerate the zeros $a_1, a_2, \cdots, a_n$ in the order in which they occur on any of these lines. Let ϖ_0 stand for the polygonal line in (p_0) for which $P_\nu = a_{\nu+1}$ $(\nu = 0, 1, \cdots, n-1)$. We are going to show that if the argument of $G(z)$ varies in a monotonic manner along all lines of (p), then ϖ_0 together with the line (a_n, a_1) encloses a convex polygon. This is more precisely stated as follows.

THEOREM III. *If* arg $G(z)$ *is never increasing* (never decreasing) *when* z *traces any line of* (p), *then the angles* arg $(a_{\nu+1} - a_\nu)$ *form a never decreasing* (never increasing) *set.*

To fix our ideas, let us assume arg $G(z)$ never increasing and put arg $(a_{\nu+1} - a_\nu) = \theta_\nu$. Then we have to prove that $\theta_\nu \leqq \theta_{\nu+1}$, for all values of ν. By (6) we have

$$(8) \qquad \begin{cases} 2\theta_\nu = -\arg \mathfrak{I}(a_\nu, a_{\nu+1}), \\ 2\theta_{\nu+1} = -\arg \mathfrak{I}(a_{\nu+1}, a_{\nu+2}). \end{cases}$$

But arg $G(z)$ is never increasing along ϖ_0, so that

$$(9) \qquad \begin{cases} \arg G(a_{\nu+1}) \leqq \arg \mathfrak{I}(a_\nu, a_{\nu+1}) \leqq \arg G(a_\nu), \\ \arg G(a_{\nu+2}) \leqq \arg \mathfrak{I}(a_{\nu+1}, a_{\nu+2}) \leqq \arg G(a_{\nu+1}). \end{cases}$$

In view of (8) and (9) we conclude that $\theta_\nu \leqq \theta_{\nu+1}$ as asserted.

3. *A Zero-free Region.* Formula (5) suggests a still more general point of view on the problem in hand. At each point z of the plane where $G(z)$ is analytic and $\neq 0$ there is a critical direction θ_z determined by

$$(10) \qquad \theta_z = -\tfrac{1}{2} \arg G(z).$$

z and θ_z define a lineal element p_z. These elements envelop a set of curves $\mathfrak{S}$ which are integral curves of the differential equation

$$(11) \qquad \mathfrak{Im}[\sqrt{G(z)}\, dz] = 0.$$

In fact, from (10) it follows that at the point z

$$(12) \qquad \tfrac{1}{2} \arg G(z) + \arg dz \equiv 0 \pmod{\pi},$$

which is equivalent to (11).

Let us take a point z_1 where $G(z)$ is analytic and $\neq 0$. Mark the singular points $a_1, a_2, \cdots, a_n, \cdots$ of $G(z)$ and join them with ∞ by means of straight lines which produced back-

wards pass through z_1. These lines $L_1, L_2, \cdots, L_n, \cdots$ we consider as cuts in the plane.

Take a pencil of rays $l(\varphi, z_1)$ with $z = z_1$ as vertex. Let us follow a special ray $l(\varphi_0, z_1)$ from z_1. Note the lineal elements along l; if $\varphi_0 \neq \theta_{z_1}$, we proceed until we arrive at a point $z_2 = z_2(\varphi_0)$ where $\theta_{z_2} \equiv \varphi_0 \pmod{\pi}$; and the remainder of the ray we leave out. From the rays that pass through the singular points we leave out at least the lines L_ν. The rays $\varphi_0 \equiv \theta_{z_1} \pmod{\pi}$ we follow until we meet a second point z_2 for which $\theta_{z_2} \equiv \varphi_0$. The points $z_2(\varphi)$ form a certain curve which may lie partly at infinity and in general also has isolated discontinuities. The finite points, except possibly the singular points a_ν, form part of an analytic curve $\mathfrak{T}(z_1)$ which is the locus of the points of tangency of the pencil of straight lines through $z = z_1$ with the family of curves $\mathfrak{S}$. From the differential equation of the pencil and of the family, namely,

$$\begin{cases} \Im\mathrm{m}[\sqrt{G(z)}\, dz] = 0, \\ \Im\mathrm{m}[dz/(z - z_1)] = 0, \end{cases} \tag{13}$$

we conclude that the equation of $\mathfrak{T}(z_1)$ is

$$\Im\mathrm{m}[\sqrt{G(z)}\ (z - z_1)] = 0. \tag{14}$$

If $z_2(\varphi)$ is discontinuous at $\varphi = \varphi_1$, we assume the existence of $z_2^- = \lim z_2(\varphi_1 - \epsilon)$ and of $z_2^+ = \lim z_2(\varphi_1 + \epsilon)$, neither of which is required to be finite. If we add the part of the ray $l(\varphi_1)$ from z_2^- to z_2^+, we make the boundary curve continuous at $\varphi = \varphi_1$. If $z_2(\varphi_\nu) = a_\nu$, one of the singular points, we have to complete $z_2(\varphi)$ by parts of the edges of the cut L_ν, namely, from a_ν to z_2^- and from a_ν to z_2^+ respectively. What we add from the cuts we consider as *inaccessible* parts of the boundary, whereas the additions from the interior of the cut plane as well as the part of $\mathfrak{T}(z_1)$ which belongs to $z_2(\varphi)$ are regarded as accessible.

The points on $l(\varphi, z)$ from z_1 to $z_2(\varphi)$ $(0 \leqq \varphi < 2\pi)$ form a region $*(z_1)$ which we shall call the *star* of z_1.* We count the accessible part of the boundary as belonging to the star;

* In a paper *Oscillation theorems in the complex domain*, to appear in the TRANSACTIONS OF THIS SOCIETY, we have given a different definition of the star. The definition given above is simpler and slightly more general.

the inaccessible points on the cuts and the *vertex* $z = z_1$ we do not include in $*(z_1)$. Then we have

THEOREM IV. *If* $W(z) = w(z)(dw/dz)$, *where* $w(z)$ *is a solution of* (1), *and if* $W(z_1) = 0$, *then* $W(z)$ *does not vanish in the star of* z_1.

The proof is obvious. Suppose z_3 is a point in $*(z_1)$. Then

$$(15) \quad (k+1)\pi > 2 \arg (z - z_1) + \arg G(z) > k\pi, \ (k=0 \text{ or } 1),$$

for all values of z on the segment (z_1, z_3) and where k only depends upon z_3. This shows that the integral on the left-hand side in formula (5) with $r_1 = |z_3 - z_1|$ cannot be positive, which is necessary if $z = z_3$ is to be a zero of $W(z)$.

If $\arg G(z)$ satisfies the restriction (2) in a convex region B, then θ_z satisfies the inequality (3); that is to say, the line part of the lineal element p_z is a line in (p) as far as the neighborhood of z is concerned. If the variation of the argument of $G(z)$ along lines in (p) is monotonic, then the parts of the curves $\mathfrak{S}$ which lie in B are all curved in the same way. In fact the curvature of a curve $\mathfrak{S}_0$ is found to be

$$(16) \quad \frac{1}{r} = -\frac{1}{2}\frac{d\phi}{ds},$$

where $\phi = \arg G(z)$ and s is the length of arc along $\mathfrak{S}_0$.* The curve $\mathfrak{S}_0$ lies to the right of the positive tangent if $(d\phi/ds) > 0$, but to the left if $(d\phi/ds) < 0$.

Assume, to fix our ideas, $(d\phi/ds) < 0$. Then the part of $\mathfrak{T}(z_1)$ that lies in B lies to the left of p_{z_1} except at $z = z_1$, which is the point of tangency. $\mathfrak{T}(z_1)$ is convex with regard to p_{z_1} and cuts the boundary of B in two points only. The points on and to the right of p_{z_1} in B belong to $*(z_1)$, and likewise those on and to the left of $\mathfrak{T}(z_1)$.

It is worth noticing that the form of the star depends only upon the argument of $G(z)$. Thus if we replace $G(z)$ by $k^2G(z)$, where k is real, we get exactly the same configuration. The frequency of the oscillation is of course changed by such a replacement.

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* The curvature is by definition $d\theta_z/ds$.

BOOKS ON FOURIER SERIES

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Second edition, revised throughout and enlarged. Vol. 1. Cambridge, the University Press, 1921. xv + 671 pp.

Introduction to the Theory of Fourier's Series and Integrals. By H. S. Carslaw. Second edition, completely revised. London, Macmillan and Company, 1921. xi + 323 pp.

The study of Fourier's series has exercised a profound influence upon the development of the theory of functions of a real variable.* Any one familiar with this influence and with the close relationship, both historical and inherent, between the two theories, will not be surprised that the two treatises mentioned above should have a considerable number of points in common. He will also not be surprised that in each case the plan of writing such a work had its origin in the study of Fourier's series in connection with their applications to problems of mathematical physics. The points of difference in scope and content between the two books are due in the main to the difference in the ultimate aim of the two writers. Professor Carslaw decided to write a book primarily for the worker in applied mathematics who has occasion to make use of Fourier's series and integrals; Professor Hobson elected to meet the needs of the pure mathematician whose work deals directly or indirectly with functions of a real variable.

A casual glance at the table of contents of Carslaw's work, however, will convince the informed reader that the needs of the pure mathematician of one generation are very apt to become the needs of the applied mathematician of the next generation. It is not so long ago that it would have been considered very unorthodox to include such topics as a discussion of Dedekind's theory of irrational numbers, the nature of uniform and non-uniform convergence, and a fairly complete treatment of the Riemann integral in a book intended for the applied mathematician. So, if the reviewer admits that Hobson's book is mainly for the pure mathematician, he does it with the mental reservation that the statement applies only to the present time and the immediate future. He does not agree with the implications contained in Hobson's statement on page 432 that the Riemann integral "will continue to be the basis upon which the practical applications of the Integral Calculus rest", but thinks it quite likely that at some future time, more or less distant, certain workers in applied mathematics may find their center of interest transferred from the Riemann integral to the Lebesgue integral, or some other integral still more general, just as at the present time some of those pursuing applied mathematics have found it desirable to change from a basis of euclidean geometry to a basis of non-euclidean geometry.

* Cf. E. B. Van Vleck, *The influence of Fourier's series upon the development of mathematics*, SCIENCE, new ser., vol. 39 (1914).

The rapidity with which the Lebesgue integral has become the standard type in present-day analysis is well exhibited by the difference in the relative emphasis placed on the Riemann and Lebesgue theories in the two editions of Hobson's book. In the first edition the latter theory was accorded a position of secondary importance; in the present edition it is placed on an equal basis, or if anything receives the greater stress. It is also illuminating in this connection to note that most of the new material included in the present edition is connected with advances concerned with or dependent on Lebesgue's theory.

As the reader will infer from the preceding paragraph, the second edition of Hobson's work takes due account of the great developments that have taken place during the past fifteen years in the theory of functions of a real variable and particularly in the theory of integration. Likewise Carslaw has not been content to rest on the laurels of his first edition, but has incorporated in his second edition many of the recent advances in the theory of Fourier's series and integrals that seem of special significance to the worker in applied mathematics. Thus both authors have aimed in the second edition as in the first to write the best possible book of the type selected on the basis of all the literature available at the time of writing. In both cases they have approached measurably near to the high ideal that they set for themselves.

Since the first editions of the two books are so widely known, the present review will not consider in great detail such portions of the second editions as are carried over from the first with relatively little change, but will concern itself mainly with the additions and the more important revisions that have been made.

Turning first to Hobson's work, we find that Chapter I is devoted to an adequate discussion of the system of real numbers, presenting both the Dedekind and the Cantor treatment of the irrationals. It is practically unchanged from the first edition except for the addition of a treatment of mathematical induction due to Padoa. The purpose of this discussion is to show that the principle referred to is a theorem which may be deduced from the properties of a simply infinite ascending aggregate by means of the principle of contradiction. This is essentially a refutation of Poincaré's contention that the principle of mathematical induction is a special characteristic of mathematical reasoning and cannot be reduced to the principle of contradiction.

Chapters II and III, which deal respectively with the descriptive properties and the metric properties of sets of points, constitute the revised form of the second chapter of the first edition. The material contained in this chapter has been much expanded in order to prepare the way for the more extensive treatment of Lebesgue integrals and their generalizations, and in numerous instances the form of the exposition has been considerably changed. One of the most extensive alterations of this sort is in connection with the treatment of non-linear sets of points. In the first edition this treatment was postponed until the exposition of the properties of linear sets had been completed. In the present edition the notion of sets of points in two or more dimensions is introduced early in Chapter II,

and the more important theorems with regard to linear sets are extended at once to sets of several dimensions after they have been proved for the case of one dimension. This change has both advantages and disadvantages. It does serve to focus the attention on the generality of the theorems involved, but on the other hand it interrupts the continuity of the argument. Opinions will probably differ as to whether or not it constitutes an improvement.

Chapter IV, which deals with transfinite numbers and order types, corresponds to Chapter III in the first edition. Relatively little change has been made in the material presented, except that the discussion of certain disputed points in the theory of assemblages, which concludes the chapter, has been revised and expanded.

Chapter V, which deals with general properties of functions of a real variable and which corresponds to Chapter IV of the first edition, contains a considerable amount of new material. This is quite natural in view of the many new developments in function theory proper during the period from 1907 to 1920. Some of this added material was available in 1907, but in most of these cases it was probably the light shed by later developments which changed the author's opinion as to its relative importance. Among the new topics discussed may be mentioned in particular the following: absolute continuity, approximate continuity, the symmetry of functional limits, functions of a variable aggregate. In connection with the latter topic, the notion of a general variable is briefly described and some of the recent developments centering about this idea are mentioned. In view of the prominent rôle of the general variable in much recent work in analysis of fundamental importance, a fuller account of it would seem desirable, even in a work devoted primarily to a very complete treatment of classical analysis.

Chapters VI, VII, and VIII are devoted to the theory of integration, and correspond to Chapter V and a small portion of Chapter VI of the first edition. This part of the work has had the greatest expansion, and this expansion is mostly in connection with the Lebesgue integral and some of its generalizations. As pointed out before, one of the chief characteristics of the present edition of the book, as contrasted with the first, is the shift in emphasis from the Riemann theory to the Lebesgue theory. The tendency in the later edition is to make this latter theory the central topic in integration and group the other developments about it. Thus, although the Riemann integral, on account of its historical priority and the somewhat greater concreteness of the geometric ideas underlying its definition, is naturally treated first, some of the important properties of the definite integral, such as the mean value theorems, are proved for the first time after the introduction of Lebesgue integrals.

In addition to a complete account of the Riemann and Lebesgue integrals, the three chapters devoted to integration contain more or less extended treatments of other types of integrals, such as the Riemann-Stieltjes integral, the Lebesgue-Stieltjes integral, Hellinger's integrals, the Denjoy integral, and the Young integral. Since the theory of integration is still in a stage of active development, it is rather too much to expect a

treatise such as Professor's Hobson's, which naturally aims to include only such researches as have reached a state of comparative completeness, to give an entirely adequate picture of the present state of the theory. Moreover, some authorities might disagree with him as to the relative emphasis he accords to the various generalizations of the notion of integration. But the reader who has perused with care the 230 odd pages devoted to this highly important topic will have a fairly satisfactory idea of the wide extension of its scope involved in the newer definitions and the much greater power of these definitions in dealing with the applications of the theory.

The reviewer has earlier expressed his high opinion of the book as a whole. It is hardly to be expected that a work of this size and scope should be free from minor defects; there are certain of these that it seems worth while to indicate. Hobson's references to the literature would have been much more valuable to the reader if he had uniformly included dates of publication in the case of all important memoirs. The almost uniform omission of such dates is hardly excusable in an account of a theory where the works of various writers have so many interrelations and the reader naturally wishes to orient himself as to the order of ideas and the reaction of one line of thought upon another. The use of the symbol $\sim$ to indicate approach to a limit also seems unwise to the reviewer. While this symbol has had some connection with approach to a limit when used in the case of asymptotic series, it has generally been used to represent correspondence in a broader sense. The symbol $\rightarrow$ is much more suggestive of the underlying notions and has been widely adopted by analysts in the time that has elapsed since its introduction.* It does not appear likely that Hobson's notation will supplant it.

Turning now to Carslaw's book, we find that the first volume of the second edition exhibits an expansion over the corresponding material in the first edition that is even greater proportionally than in the case of Hobson's book. This will not be surprising to those familiar with recent developments in the theory of Fourier's series and integrals, for many of these developments have been of special significance to those primarily interested in the applications of the theory. The new material added by Carslaw is not exclusively composed of such new developments, however. A considerable portion of it is devoted to a more elaborate development of certain topics of pure analysis that are of fundamental importance for those who seek to use in any rational manner such tools as Fourier's series and integrals in the field of applied mathematics.

The book begins as before with an interesting historical introduction, which in the present edition has been somewhat revised and expanded. Chapter I, devoted to an exposition of the system of real numbers, and Chapter II, dealing with the more fundamental properties of infinite sequences and series, contain substantially the same material as in the

* It may be worthy of note in this connection that this BULLETIN has adopted this notation from the beginning of the present volume. (The editors.)

first edition, though they have been considerably revised. Chapter III of the present edition is new and contains a development of the notions of function and limit and some of the more fundamental properties of functions of one or more variables. The addition of this chapter is in line with a suggestion by the late J. E. Wright in the course of his review of the first edition in this BULLETIN.

Chapter IV deals with the same material as the first part of Chapter IV in the first edition, but the material has been much expanded and revised. An adequate treatment of the Riemann integral and its more important properties has been included in this chapter. Chapter V is the original Chapter III in revised form and with a considerable amount of new material at the end. This includes a proof of Bromwich's theorem on convergence factors which is of importance in many applications of Fourier's series and allied developments, and a discussion of the integration of infinite series. Chapter VI contains the remaining portion of Chapter IV, namely, the treatment of definite integrals involving a parameter. This portion also has been revised and expanded.

Chapter VII corresponds to the original Chapter V and is devoted to the discussion of the more fundamental theorems with regard to the convergence and summability (C1) of the Fourier development of an arbitrary function. Among the new material added may be mentioned the proof of the convergence of Fourier's series by means of Fejér's theorem on the summability (C1) and a general theorem with regard to series summable (C1) due to G. H. Hardy; also a discussion of Poisson's integral.

Chapter VIII, dealing mainly with the uniform convergence of Fourier's series and the validity of term by term differentiation and integration of such series, and Chapter IX, devoted to the consideration of the approximation curves and Gibb's phenomenon, correspond to Chapters VI and VII of the original edition. The material on Gibb's phenomenon is new and is founded mainly on Bôcher's discussion in the ANNALS OF MATHEMATICS and the further treatment of the subject by the author in the AMERICAN JOURNAL OF MATHEMATICS.

Chapter X, which deals with Fourier's integrals, corresponds to the original Chapter VIII. In its present form it includes recent extensions of the theory of these integrals due to Pringsheim. Appendix I, which is new, gives a brief account of certain methods of harmonic analysis that are useful when the function to be developed is not given in analytic form but only by means of a curve obtained from observations. Appendix II contains the bibliography of Appendix A in the first edition, brought up to date. This bringing up to date involves the addition of seven and a half pages of titles for the period 1906–1920.

CHARLES N. MOORE

SHORTER NOTICES

Oeuvres de G. H. Halphen. Publiées par les soins de C. Jordan, H. Poincaré, É. Picard, avec la collaboration de E. Vessiot. Tome III.* Paris, Gauthier-Villars, 1921. xii + 518 pp.

The Paris Academy of Sciences proposed as the subject of the grand prize in mathematics for 1880 the following question: "To perfect in some important part the theory of ordinary linear differential equations." Halphen submitted a memoir in which he proposed to determine those ordinary linear differential equations which are reducible to integrable forms by means of transformations of a certain class; and the prize was awarded to this memoir. The text of Hermite's report on the memoir is reproduced in an article on Halphen published by C. Jordan in LIOUVILLE'S JOURNAL in 1889 and reprinted on pages v to xii of the volume under review. (See also p. 33 of vol. I of Halphen's *Oeuvres*.)

Halphen formulated in the following manner the exact "double question" which he treated:

(1) Having given a linear differential equation with variable X and unknown Y, to determine if there exists a substitution

$$x = \varphi(X), \qquad y = Y\psi(X),$$

such that, x and y being taken for the new variable and the new unknown, the transformed equation belongs to one of the following three categories:

I. Equations with constant coefficients;

II. Equations whose general integral is rational;

III. Equations with doubly periodic coefficients with the same periods and with a uniform general integral.

(2) Having determined the existence of the substitution, to effect the integration.

The memoir in which he treats this question is reprinted on pages 1–260 of the volume under review. It opens with an excellent fourteen-page summary of the results attained. The response to the first question is entirely decisive: necessary and sufficient conditions are obtained (and summarized in the introduction) for the existence of the substitution named of such sort that one may apply them to a particular equation by means of straightforward reckoning. The propositions relating to the second part are not quite so definitive; but they are far-reaching enough to have important applications, particularly to equations of the second order.

The judgment of the Academy, awarding the grand prize to Halphen for this memoir, was scarcely rendered when the Berlin Academy proposed for the Steiner prize "the solution of an important question in the theory of twisted algebraic curves." In 1870 Halphen had published a short note on the classification of these curves. He now returned to this subject and produced his classic memoir on the classification of twisted algebraic curves (reprinted on pp. 261–455 of the volume under review). It was crowned conjointly with a memoir by Noether submitted on the same

* For a short review of volumes I and II see this BULLETIN, vol. 27 (1920–21), pp. 466–468.

occasion. By many this is esteemed to be Halphen's chief work. Some account of it is given in the review of volumes I and II in connection with Poincaré's analysis of the memoir on pages xxx–xxxi of volume I.

Besides these two master memoirs, volume III of the *Oeuvres* contains the introductory article by C. Jordan already referred to, and two articles by Halphen closely related to the former of the two already treated. They both deal with linear differential equations of the fourth order: one is a brief note (pp. 457–461) from the COMPTES RENDUS and the other is a longer article (pp. 463–514) from the ACTA MATHEMATICA on the invariants of linear differential equations of the fourth order.

The four articles reprinted in this volume may justly be said to contain two great contributions by Halphen. The one is the memoir on twisted algebraic curves and the other is the work on linear differential equations contained in the first, third and fourth papers. These are all entirely inspired by the theory of differential invariants (to which Halphen had previously made characteristic contributions); and they are treated together by Halphen himself (vol. I, pp. 32–35) in the summary of his work prepared by him on the occasion of his candidacy before the Paris Academy in 1885.

R. D. CARMICHAEL

Graphical and Mechanical Computation. By Joseph Lipka. New York, John Wiley and Sons, 1918. ix + 264 pp.

This book is something new in American text-books. It treats two subjects of vital importance to engineering students that heretofore have found little place in the curricula of American engineering schools. The first part of the book is devoted to the graphic solution of engineering problems by means of networks of scales, various kinds of coordinate paper, the charting of equations in three variables and more particularly by means of alignment charts. The second half of the book is concerned with the question of empirical formulas for both non-periodic and periodic curves giving numerical, graphic and mechanical methods for determining the constants. For data which cannot be fitted to convenient formulas, numerical, graphic and mechanical methods are developed in the last two chapters for interpolation, differentiation and integration.

The book is rich in illustrations and applications to practical engineering problems, showing clearly the value of the graphic methods in reducing the drudgery of long computations. The three chapters on alignment charts are recommended to the student who wishes to know something of the nomographic charts of Professor M. d'Ocagne. The work is also published in two volumes.

A. R. CRATHORNE

Funktionentheorie. By Konrad Knopp. 2d edition. Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger. Part I, 1918, 140 pp. Part II, 1920, 138 pp.

This "Funktionentheorie" is a completely revised edition of the work that appeared under the same title in 1913 in the Sammlung Göschen. It is issued in a form similar to the old one by the Vereinigung wissenschaftlicher Verleger which, since the war, has continued the publishing

activities of several houses, among them the Göschen'sche Verlagshandlung.

The book, grouping its material around the centrally important concept of the monogenic analytic function, affords a very satisfactory brief introduction to the study of the theory of functions of a complex variable. By maintaining throughout sufficient emphasis upon the central concept, and by connecting various parts of the theory by their relation to this concept, a balance that is very desirable in a book of this kind is secured between the more and the less theoretical aspects of the subject. After a chapter on point sets, *Wege* (i.e., curves consisting of a finite number of rectifiable pieces), etc., there are taken up the integral theorems, development in series, analytic continuation and singular points. Then follow, in the second volume, short sections on entire functions, meromorphic functions and periodic functions (with brief mention of the elliptic functions); and finally a section on multiple-valued functions.

The book is clearly written, well printed, and practically free from misprints. The Cauchy integral theorem is proved for a closed *Weg* by means of approximating polygonal lines and by reducing the theorem ultimately to the case in which the area enclosed by the curve is a triangle, in a manner similar to that followed by Jordan in volume I of his *Cours d'Analyse*. A number of well-chosen exercises are distributed throughout the book.

ARNOLD DRESDEN

Réflexions sur la Métaphysique du Calcul Infinitésimal. By Lazare Carnot. I and II. Paris, Gauthier-Villars, 1921.

It is not generally known that this treatise was translated into English by William Dickson and published, with notes, in the PHILOSOPHICAL MAGAZINE of London, volumes 9 and 10, for the years 1800 and 1801. Maurice Solovine, the editor of the present edition, is mistaken when he states that J. K. Hauff's German translation (1800) and G. B. Magistrini's Italian translation (1803) antedated translation into English. To be sure, a later English version, prepared by William Robert Browell, did appear at Oxford in 1832. The various translations bear testimony to the high esteem in which the booklet was held by Carnot's contemporaries. Carnot makes no reference to English mathematicians of the eighteenth century who came after Newton; he was unacquainted with Bishop Berkeley's onslaught as found in the ANALYST. Berkeley and Carnot exerted a strong influence on the development of the philosophy of the calculus: the one by his destructive criticism, the other by his constructive efforts. Carnot lays great stress on the doctrine of the compensation of errors in infinitesimal analysis, being unaware that before him this view-point had been presented by Berkeley and others. While D'Alembert and Lhuilier were partial to the method of limits, Carnot found that method a tortuous foot-path in which it was difficult to avoid being bewildered. Mathematicians interested in the history and philosophy of the calculus will welcome this reprint of the greatly enlarged edition of 1813. It appears in the series of inexpensive editions of *Les Maîtres de la Pensée Scientifique.*

FLORIAN CAJORI

NOTES

On account of certain regulations of the United States Postoffice Department, the April–May number of this BULLETIN was given only one serial number, though it is in fact a double number. It is number 4 of volume 28, and number 304 in the total numbering. For this reason, the present volume will contain only nine numbers according to this serial numbering, but there will be issues for each of ten months, as usual, and the total number of pages in the volume will remain at approximately 500 pages, as usual. The only irregularity will be in the serial number of the April–May issue, as explained above.

On April 18, a celebration in honor of Professor Charlotte Angas Scott was held at Bryn Mawr College by her former students. (See this BULLETIN, vol. 28, p. 72.) In the afternoon, after an address of welcome by President M. Carey Thomas and an appreciation of Professor Scott as a teacher and an investigator by Miss Marion Reilly, Professor A. N. Whitehead gave an address on *Some principles of physical science*. The lecture was followed by a tea for all the guests at the Deanery, the home of President Thomas. At a dinner in the evening, there were present about seventy members of the American Mathematical Society, about seventy former students, and a number of members of the Faculty of the College. Representatives of each of these groups gave short talks testifying to Professor Scott's early brilliant record in England, to the inspiration and encouragement that her research and career have brought especially to women beginning graduate work in mathematics, and to the invaluable services that she renders to the College by her teaching, her committee work, and her council.

The October number (vol. 43, No. 4) of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *On some properties of general manifolds relating to Einstein's theory of gravitation*, by J. A. Schouten and D. J. Struik; *Geometrical theorems on Einstein's cosmological equations*, by Edward Kasner; *On the Fermat and Hessian points for the non-euclidean triangle and their analogues for the tetrahedron*, by C. M. Sparrow; *The Cauchy-Lipschitz method for infinite systems of differential equations*, by W. L. Hart; *Boundary value and expansion problems; formulation of various transcendental problems*, by R. D. Carmichael; *Reciprocity in a problem of relative maxima and minima*, by J. K. Whittemore.

Professor Gino Loria announces that the BOLLETTINO DI BIBLIOGRAFIA E STORIA DELLE SCIENZE MATHEMATICHE, which he founded and of which he has been editor for many years, will in future be published as an historical and bibliographical section of the BOLLETTINO DI MATEMATICA, of which Dr. A. Conti is editor. It is proposed to retain all the essential features of Professor Loria's BOLLETTINO under the new form.

At the meeting of the National Academy of Sciences held at Washington April 24–26, 1922, the following mathematical papers were read: *Some extensions in the mathematics of hydro-mechanics*, by R. S. Woodward; *Normal coordinates and Einstein space*, by G. D. Birkhoff; *Algebraic solutions of Einstein's cosmological equations*, by Edward Kasner; *The geometry of paths*, by Oswald Veblen. Professor H. A. Lorentz, of the University of Leiden, delivered the evening address on April 24, at the invitation of the Academy and the Carnegie Institution of Washington. Professor L. P. Eisenhart was elected a fellow of the Academy, and Professor Albert Einstein a foreign associate.

The following fourteen doctorates with mathematics as major subject were conferred by American universities in the academic year 1920–21; the title of the dissertation is added in each case: Nina M. Alderton, California, *Involutory quartic transformations in space of four dimensions;* Beulah M. Armstrong, Illinois, *Mathematical induction in group theory;* E. M. Berry, Iowa, *Diffuse reflection,* Rachel Blodgett, Radcliffe, *Determination of the coefficient in interpolation formulæ and a study of the approximate solution of integral equations;* P. H. Daus, California, *Normal ternary fraction expansions for the cube roots of integers;* W. E. Edington, Illinois, *Abstract group definitions and applications;* M. C. Foster, Yale, *Rectilinear congruences referred to special surfaces;* Philip Franklin, Princeton, *Four color problem;* Mayme I. Logsdon, Chicago, *Equivalence and reduction of pairs of hermitian forms;* Irwin Roman, Chicago, *Transformation of waves through a symmetrical optical instrument;* D. V. Steed, California, *Lines on the hypersurface of order* $2n - 3$ *in space of n dimensions;* Jung Sun, Syracuse, *Some determinant theorems;* Flora D. Sutton, Johns Hopkins, *Certain chains of theorems in reflective geometry;* F. E. Wood, Chicago, *Certain relations between the projective theory of surfaces and the projective theory of congruences.*

The Carl Zeiss Foundation of Jena announced, at the meeting of mathematicians and physicists held at Jena in September, 1921, the establishment of a prize in memory of Ernst Abbe, "for the advancement of the mathematical and physical sciences and their applications." The interest on a fund of 100,000 marks will be awarded every two years for important research in these fields. The prize will be awarded for work in mathematics in 1924, in physics in 1926, and in applied mathematics and physics in 1928.

Cambridge University has awarded a Smith's prize to E. A. Milne, of Trinity College, for an essay entitled *Studies in the theory of radiative equilibrium.*

Professor René Baire has been elected correspondent of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor M. Noether.

Professor A. S. Eddington has been elected a foreign honorary member of the American Academy of Arts and Sciences.

The following advanced courses in mathematics are announced for the academic year 1922–23:

COLUMBIA UNIVERSITY.—By Professor T. S. Fiske: Differential equations (first term).—By Professor F. N. Cole: Algebra.—By Professor D. E. Smith: History of mathematics; Practicum in the history of mathematics.—By Professor C. J. Keyser: Introduction to mathematical philosophy (first term); Logical foundations of mathematics.—By Professor Edward Kasner: Einstein's theory of gravitation.—By Professor W. B. Fite: Theory of functions.—By Professor J. F. Ritt: Functions of several complex variables (first term); Algebraic numbers (second term).—By Dr. G. A. Pfeiffer: Isoperimetric problems (second term).—By Dr. Jesse Douglas: Differential geometry (first term).

CORNELL UNIVERSITY.—By Professor J. I. Hutchinson: Entire functions. By Professor Virgil Snyder: Algebraic geometry.—By Professor F. R. Sharpe: Vector analysis.—By Professor W. B. Carver: Advanced calculus.—By Professor Arthur Ranum: Differential geometry.—By Professor D. C. Gillespie: The definite integral.—By Professor W. A. Hurwitz: Infinite series.—By Professor C. F. Craig: Probabilities.—By Professor F. W. Owens: Projective geometry.—By Professor H. M. Morse: Einstein's theory (first term); Dynamical systems (second term).—By Professor W. L. G. Williams: Modern higher algebra.—By Dr. F. W. Reed: Elementary differential equations.—By Mr. H. S. Vandiver: Finite groups.—By Dr. G. M. Robison: Advanced analytic geometry.

HARVARD UNIVERSITY.—All courses meet throughout the year, except those marked with an asterisk, which meet for half a year.—By Professor W. F. Osgood: Differential and integral calculus (advanced course); Theory of functions (introductory course).—By Professor J. L. Coolidge: Probability*; Algebra*; Algebraic plane curves.—By Professor E. V. Huntington: The fundamental concepts of mathematics*.—By Professor O. D. Kellogg: Dynamics (second course); Introduction to the theory of potential functions and Laplace's equation*; Potential functions (advanced course)*.—By Professor G. D. Birkhoff: The analytic theory of heat and problems in elastic vibrations*; Linear differential equations of the second order, real variables*.—By Professor W. C. Graustein: Introduction to modern geometry; Differential geometry of curves and surfaces.—By Dr. J. L. Walsh: Infinite series and products*; Theory of numbers*; Entire functions*.—By Dr. Philip Franklin: Elementary theory of differential equations*; Analysis situs*.—Dr. Walsh and Dr. Franklin will conduct a fortnightly seminar in analysis. Courses of research are offered by Professor Osgood in the theory of functions, by Professor Coolidge in geometry, by Professor Kellogg in the theory of potential functions, by Professor Birkhoff in the theory of differential equations, and by Professor Graustein in geometry.

Professor Alexander Ziwet has presented his private library, consisting of more than five thousand books on mathematics and mechanics, to the University of Michigan. This important collection includes many rare works.

At the University of Strasbourg, Dr. E. Bauer, maître de conférences, has been promoted to a professorship of mathematics.

Mr. W. H. Durfee, of Harvard University, has been appointed assistant professor of mathematics at Hobart College.

Mr. J. A. Northcott, of Columbia University, has been promoted to an assistant professorship of mathematics, in charge of extension courses.

Dr. C. A. Nelson, of Western Reserve University, has been appointed associate in mathematics at Johns Hopkins University.

At the University of Minnesota, Professor A. L. Underhill has leave of absence for the year 1922–23; he expects to spend the year in France.

The following appointments to instructorships in mathematics are announced: Cornell University, Mr. W. W. Elliott; Harvard University, Dr. Philip Franklin (Pierce instructor), Mr. K. W. Holbert, Mr. H. C. Shaub, Mr. M. M. Slotnick, and Mr. D. V. Widder; University of Minnesota, Mr. H. W. Chandler.

Professor D. E. Smith, vice-president of the Society, and Mrs. Smith were seriously injured in an automobile accident on May 30 in Northern France. Both recovered after being in a hospital for over a month. They will return to America in September.

Sir A. B. Kempe, president of the London Mathematical Society in 1894, and for many years treasurer of the Royal Society, died April 27, 1922, at the age of seventy-three years.

Professor J. B. Coit, of Boston University, died July 26, 1921, at the age of seventy-two years.

Dr. G. B. Halsted, professor of mathematics at the University of Texas from 1882 to 1903, and later at St. John's College, Kenyon College, and the Colorado State Teachers' College, died in New York City March 19, 1922, at the age of sixty-nine years. Dr. Halsted had been a member of the American Mathematical Society from the time of its organization, in 1891, as the New York Mathematical Society. He was the author of several books and papers on non-euclidean geometry, and on the foundations of geometry.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

ANDOYER (H.). L'œuvre scientifique de Laplace. Paris, Payot, 1922. 16mo. 162 pp.

BERKHAN (G.). See ENCYKLOPÄDIE.

BIANCHI (L.). Lezioni di geometria differenziale. 3a edizione interamente rifusa ed ampliata, in due volumi. Volume 1. Pisa, Spoerri, 1921. 8vo. 800 pp.

DUPORT (—.). Les transformations de Backlund. Paris, Gauthier-Villars, 1921. 8vo. 10 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 1, Heft 7: G. Berkhan und W. F. Meyer, Neuere Dreiecksgeometrie. H. Rothe, Systeme geometrischer Analyse, 1ter Teil. Leipzig, Teubner, 1921.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 3, Heft 5: E. Salkowski, Dreifach-orthogonale Flächensysteme. Leipzig, Teubner, 1921.

FARKAS (A.). Die Auflösung einiger unbestimmten Gleichungen höheren Grades durch Reihen. Nagyenyed, Aiud, 1920. 7 pp.

GEIRINGER (H.). Die Gedankenwelt der Mathematik. Berlin und Frankfurt, Verlag der Arbeitsgemeinschaft, 1922. 200 pp.

HAMPEL (H.). Ueber Aoustsche Kurven. (Diss., Halle-Wittenberg.) Halle, 1921. 86 pp.

HILBERT (D). Die Grundlagen der Geometrie. 5te Auflage. (Wissenschaft und Hypothese, Nr. 7.) Leipzig, 1922.

JOHNSON (W. E.). Logic. Part 2: Demonstrative inference, deductive and inductive. Cambridge, University Press, 1922. 20 + 258 pp.

KOWALEWSKI (A.). Die Buntordnung. Mathematische, philosophische und technische Betrachtungen über eine neue kombinatorische Idee. Heft 1. Leipzig, Engelmann, 1922. 53 pp.

MEYER (W. F.). See ENCYKLOPÄDIE.

NEVILLE (E. H.). Multilinear functions of direction, and their uses in differential geometry. Cambridge, University Press, 1921. 8vo.

POIRÉE (J.). Précis d'arithmétique. Paris, Gauthier-Villars, 1921. 8vo. 6 + 64 pp.

ROTHE (H.). See ENCYKLOPÄDIE.

RYDE (F.). A contribution to the theory of linear homogeneous geometric difference equations (*q*-difference equations). (Diss., Lund.) Lund, 1921. 45 pp.

SALKOWSKI (E.). See ENCYKLOPÄDIE.

SCHEBEN (G.). Brennpunkte und Asymptoten der Kegelschnitte in der nicht-euklidischen Geometrie. (Diss.) Bonn, 1921.

WEATHERBURN (C. E.). Elementary vector analysis. London, Bell, 1921. 28 + 184 pp.

II. ELEMENTARY MATHEMATICS

AUBERT (P.) et Papelier (G.). Exercices d'algèbre, d'analyse et de trigonométrie. Tome 1. Paris, Vuibert, 1921. 366 pp.

DICKSON, (L. E.). Plane trigonometry. New York, Sanborn, 1922. 19 + 176 + 35 pp.

DINGELDEY (F.). Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. Teil 2: Aufgaben zur Anwendung der Integralrechnung. 2te Auflage. Leipzig, 1920.

DOUB (A. V.). See Ray (H. B.).

HALL (W. S.). Elements of the differential and integral calculus. 2d edition, revised. New York, 1922. 263 pp. $2.75

MOREUX (T.). Pour comprendre l'arithmétique. Paris, Doin, 1921. 16mo. 212 pp.

OBERG (E.). Solution of triangles. A treatise on the use of formulas and the practical application of trigonometry and logarithms in the solution of shop problems involving right-angled and oblique-angled triangles. New York, 1921. 100 pp. $1.00

PAPELIER (G.). See AUBERT (P.).

RAY (H. B.) and DOUB (A. V.). Preparatory mathematics for use in technical schools. New York, Wiley, 1921.

SCHMIDT (H.). Zahl und Form. Leichtfassliche Einführung in die Mathematik. Hamburg, 1921.

WILHELM (J.). Comprehensive tables of compound interest. London, Effingham Wilson, 1922. 112 pp.

III. APPLIED MATHEMATICS

AMOROSO (L.). Lezioni di matematica finanziaria. Volume 1. Napoli, Majo, 1921.

BARUS (C.). Displacement interferometry applied to acoustics and to gravitation. (Carnegie Institution Publication, No. 310.) Washington, Carnegie Institution, 1921. 8 + 149 pp. $2.50

BIRD (J. M.). See SCIENTIFIC AMERICAN.

BROSE (H. L.). See WEYL (H.).

CHAMBERS (G.). Studies in the dynamics of the earth's atmosphere. London, Harrison, 1920. 28 pp.

DINGLE (H.). Relativity for all. London, Methuen, 1922. 18mo. 8 + 172 pp.

ECCLES (W. H.). Continuous wave wireless telegraphy. Part 1. London, Wireless Press, 1921. 7 + 407 pp. 25s.

GRÄTZ (L.). Der Atombau in ihrer neuesten Entwicklung. Sechs Vorträge. Stuttgart, 1921.

HAUSSNER (R.). Darstellende Geometrie. 2ter Teil. 2te, verbesserte und vermehrte Auflage. Berlin, 1921.

KEYNES (J. M.). A treatise on probability. London, Macmillan, 1921. 11 + 466 pp.

LAMOTTE (M.). Cours de mécanique appliquée. Paris, Gauthier-Villars, 1922. 8vo. 214 pp.

LIPKA (J.). Graphical and mechanical computation. Part 2: Experimental data. New York, Wiley, 1921. 145 pp. $2.00

LORING (F. H.). Atomic theories. London, Methuen, 1921. 9 + 218 pp. 12s. 6d.

MAILLARD (L.). Le mouvement quasi newtonien et la gravitation. Lausanne, Imprimeries réunies, 1921.

MIE (G.). Die Einsteinsche Gravitationstheorie. Versuch einer allgemein verständlichen Darstellung der Theorie. Leipzig, Hirzel, 1921. 4 + 67 pp.

NERNST (W.). Das Weltgebäude im Lichte der neueren Forschung. Berlin, Springer, 1921. 2 + 63 pp.

D'OCAGNE (M.). Traité de nomographie. Paris, Gauthier-Villars, 1921. 24 + 484 pp.

PAGE (L.). An introduction to electrodynamics from the standpoint of the electron theory. Boston, Ginn, 1922. 6 + 134 pp.

PETIT (E.). See RICHARD (P. J.).

PICARD (E.). La théorie de la relativité et ses applications à l'astronomie. Paris, Gauthier-Villars, 1922. 16mo. 27 pp.

PRINGSHEIM (P.). Flouoreszenz und Phosphoreszenz im Lichte der neueren Atomtheorie. Berlin, Springer, 1921. 8 + 202 pp.

RHODES (E. C.). Smoothing. (Tracts for Computers, No. 6.) London, Cambridge University Press, 1921. 60 pp. 3s. 9d.

RICHARD (P. J.) et PETIT (E.). Théorie mathématique des assurances. 2e édition, entièrement remaniée et mise à jour. 2 volumes. (Encyclopédie Scientifique.) Paris, Doin, 1922. 455 + 320 pp.

RICHTER (H.). Die Entwicklung der Begriffe "Kraft, Stoff, Raum, Zeit" durch die Philosophie mit Lösung des Einstein Problems. Leipzig, Hillmann, 1921.

SÄNGERWALD (R.). Beiträge zur streckenweisen Errechnung von Geschossbahnen. Leipzig, Teubner, 1921.

SCIENTIFIC AMERICAN. Relativity and gravitation, by various writers. Essays written for the SCIENTIFIC AMERICAN's competition, edited by J. M. Bird. London, Methuen, 1921. 14 + 345 pp. 8s. 6d.

THIRRING (H.). Die Idee der Relativitätstheorie. Berlin, Springer, 1921. 4 + 170 pp.

TILGHER (A.). Relativisti contemporanei. 3a edizione. Roma, Libreria de Scienze e Lettere, 1921. 81 pp.

UHLICH (R.). Untersuchungen zur Erklärung der Fernwirkenden Kräfte. Leipzig, Teubner, 1920.

VILLAT (H.). Aperçus théoriques sur la résistance des fluides. Paris, Gauthier-Villars, 1921. 8vo. 101 pp. Fr. 8.00

VOLTERRA (V.). The flow of electricity in a magnetic field. Berkeley, University of California Press, 1921. 8vo. 72 pp. $1.25

WEYL (H.). Space, time, matter. Translated by H. L. Brose. London, Methuen, 1922. 8vo.

WHEELER (S. G.). Entropy as a tangible conception. London and New York, 1921. 76 pp. $3.00